PETAL GRAPHS CLASSIFICATION
BASED ON VIZING’S THEOREM

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ABSTRACT. Classifying graphs into two classes, class 1 and class 2, is difficult problem. Many scientists have made it simpler by making some conjectures and theorems. This research classify the graphs that have maximum degree three, minimum degree two, core of the graphs is 2-regular and all edges of the graphs incident with at least one vertex in core. It is named by petal graphs. By proving a theorem, they all can be classified into class 1 with a single exception: petal graph that isomorphic with Petersen graph that is deleted one vertex. Before classify the graphs, this research introduces petal size and some examples of petal graphs.

Key words: classes of graphs, petal size, petal graphs, Petersen graph

1. INTRODUCTION

This research is based on the journal written by Cariolaro and Cariolaro [2]. The aims of this research are introducing petal size and petal graphs, then classify them into classes based on Vizing’s theorem. All graphs in this research are simple, finite, connected, and undirected.

For nonempty set of vertices $V'$ subset $V$, the subgraphs which the vertices are in $V'$ and the edges are in $G$ which both ends are in $V'$ is called by subgraph of $G$ induced by $V'$ and written by $G[V']$ [1]. The term core of $G$ in this research means the subgraph induced by vertices having degree $\Delta(G)$, in which $\Delta(G)$ is maximum degree of $G$. The core of $G$ is denoted by $G_\Delta$ [2]. The distance between $v_1$ and $v_2$, for $v_1$ and $v_2$ are two vertices in $G_\Delta$, is the length of shortest $v_1 - v_2$ path in $G_\Delta$ if the path exist in $G_\Delta$. It is denoted by $dist_{G_\Delta}(v_1, v_2)$. If there are no $v_1 - v_2$ path in $G_\Delta$ then the distance is infinite [3].

There are two kinds of coloring of graphs: vertex coloring and edge coloring. Because this research studies about edge coloring, the term coloring and edge coloring be used as synonym. Wilson and Watkins [4] wrote that a $k$-edge coloring of $G$ is an assignment of $k$ colors to the edges of $G$ in such a way so that any edges which adjacent are assigned different colors. If $G$ has a $k$-edge coloring then $G$ is said to be $k$-edge colorable. The chromatic index of $G$, denoted by $\chi'(G)$, is the smallest number $k$ for which $G$ is $k$-edge colorable. If all edges of $G$ which adjacent are assigned different colors, then the
coloring is said to be proper. Carliaro and Carliaro [2] defined a (proper) k-edge coloring of a graph G as a map \( \varphi : (G) \to \mathcal{C} \), where \( |\mathcal{C}| = k \) and \( \varphi(e_1) \neq \varphi(e_2) \) for each pair \((e_1, e_2)\) of adjacent edges of G. This research use this notion (and symbol) of coloring rather than the notion written by Wilson and Watkins.

Harary [6] wrote that graph G is called critical if \( \chi'(G - v) < \chi'(G) \) for all v. The notation \( G - v \) means graph G minus (deleting) one vertex v. Of course, if G critical, then \( \chi'(G - v) = \chi'(G) - 1 \).

The classification of graphs of this research is based on Vizing’s theorem which is given below.

**Theorem 1.1.** For any simple graphs G having maximum degree \( \Delta(G) \),

\[
\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.
\]

From Theorem 1.1, graphs are classified into two classes, class 1 and class 2. A graph G is classified into class 1 if \( \chi'(G) = \Delta(G) \). And it is classified into class 2 if \( \chi'(G) = \Delta(G) + 1 \) [8].

2. PETAL GRAPHS

Since classifying graphs into class 1 or class 2 is difficult problem, many scientists tried to make it simpler. Carliaro and Carliaro [2] classified some graphs having the same properties into class 1 and class 2. The graph is named by petal graph because its shape is symmetrical like petal [5], and the following is its definitions.

**Definition 2.1.** A petal graph is a connected graph G such that

1. maximum degree of G, \( \Delta(G) = 3 \), and minimum degree, \( \delta(G) = 2 \),
2. core of G, \( G_\Delta \), is 2-regular,
3. every edge of G is incident with at least one vertex in \( G_\Delta \).

The examples of petal graphs are shown on Figure 1. They are have maximum degree three and minimum degree two. The core of each graphs is 2-regular, and every edge of the graphs is incident with at least one vertex in its core. It shows that all graphs in the figure are petal graphs.

3. PETAL SIZE

If G is a petal graph and w is a vertex of G having degree two, and the vertices which are adjacent with it are \( v_1 \) and \( v_2 \), then the path \( P_w = v_1, w, v_2 \) is a petal of G. The vertex w is named by center while \( v_1 \) and \( v_2 \) are named by basepoints. If \( \text{dist}_{G_\Delta}(v_1, v_2) = k \), then the size of petal \( P_w \) is k or \( P_w \) is k-petal. Similarly with the definition of distance, the size of petal is infinite if there is no \( v_1 - v_2 \) path in \( G_\Delta \). The petal size of G is the minimum size of petals of G. It is denoted by \( p(G) \).

In Figure 1, the graph G has two petals. They are \( P_{v_6} = v_1, v_6, v_5 \) and \( P_{v_8} = v_2, v_3, v_4 \). They both are 1-petal because \( \text{dist}_{G_\Delta}(v_1, v_5) = \text{dist}_{G_\Delta}(v_2, v_4) = 1 \) so that \( p(G) = 1 \).

The graph H has three petals: \( P_{v_6} = v_1, v_8, v_3, P_{v_9} = v_2, v_9, v_5 \), and \( P_{v_7} = v_4, v_7, v_6 \). Their size is 2, 3 and 2, respectively. Since the minimum size of petal is 2, \( p(H) = 2 \).
The graph $P^*$ has three petals and their size is all 3, so that $p(P^*) = 3$. This graph obtained from Petersen graph that is deleted one vertex. Petersen graph is defined by Weisstein [7] as a graph possessing ten nodes, all of whose nodes have degree three. It is usually represented as like Figure 2.

The petals of graph $I$ have basepoints that is not connected by a path in $I_\Delta$. It means that their size is infinite, so that $p(I) = \infty$.

4. SOME USEFUL LEMMAS

The lemmas that useful in this research are given below. They are adapted from the journal written by Cariolaro and Cariolaro [2].

**Lemma 4.1.** Let $G$ be a connected class 2 graph with $\Delta(G) \leq 2$. Then
(1) graph $G$ is critical,
(2) minimum degree of $G_\Delta$, $\delta(G_\Delta) = 2$,
(3) minimum degree of $G$, $\delta(G) = \Delta(G) - 1$,
(4) every vertex of $G$ adjacent with at least one vertex in $G_\Delta$.

**Lemma 4.2.** Let $G$ be a connected class 2 graph with $\Delta(G) = 3$ and $\Delta(G_\Delta) \leq 2$. Then $G$ is a petal graph.

**Lemma 4.3.** Let $L_n$, for any positive integer $n$, denote the graph obtained form a path $v_0, v_1, \ldots, v_{n-1}, v_n$ by inserting a path $v_i, v_j, y_i$ at each of the inner vertices $v_1, \ldots, v_{n-1}$. Let $f_i = w_i, y_i$ and let $\varphi : \{v_0v_1, f_1, f_2, \ldots, f_{n-1}\} \rightarrow \{\alpha, \beta, \gamma\}$ be an arbitrary assignment of colors. Let $\theta \in \{\varphi(f_{n-1})\}$. Then $\theta$ can be extended to a proper coloring $\hat{\varphi} : E(L_n) \rightarrow \{\alpha, \beta, \gamma\}$. Moreover such a coloring can be chosen in order to satisfy the additional requirement that $\hat{\varphi}(v_{n-1}v_n) \neq \theta$.

## 5. CLASSIFICATION

In this section, petal graphs are classified into class 1 and class 2. Cariolaro and Cariolaro wrote a theorem that is very useful for it.

**Theorem 5.1.** Let $G$ be a petal graph, and $G \neq P^*$. Then $G$ is class 1.

In this case, $P^*$ is the graph obtained by deleting one vertex of Petersen graph. It’s shown in Figure 1. Since it theorem can be proved, the classification is finish. In this research, this theorem is called ”Main Theorem”.

Before proving the Main Theorem, there are some lemmas that are proved first. The lemmas are about petal graphs whose petal size are 1, 2, and infinite. It states that they are all class 1.

**Lemma 5.2.** Let $G$ be a petal graph such that $p(G) = 1$. Then $G$ is class 1.

**Proof.** It is proved by contradiction, so that suppose that $G$ is class 2. According to Lemma 4.1, $G$ is critical. Let $P_w = v_1, w, v_2$ be a petal of $G$ such that its size is 1, and $u_1v_1, u_2v_2$ are two edges which adjacent with the edge $v_1v_2$ in $G_\Delta$. It is illustrated on Figure 3.

Let $G_1 = G - w - v_1v_2$. It is 3-colorable. Then, let $G^*$ be the graph obtained from $G_1$ by merging vertices $v_1$ and $v_2$. Named the vertex as $v^*$. Consider there is one-to-one correspondence between coloring of $G^*$ and $G_1$. It is clear that $G^*$ is not petal graph but connected, $\Delta(G^*) = 3$ and $\Delta(G^*_{\Delta}) \leq 2$. By Lemma 4.2, $G^*$ is class 1.

![Diagram](image-url)

**Figure 3.** Illustration of proof of Lemma 5.2
In $G^*$, the edge $u_1v^*$ adjacents with $u_2v^*$, so that both of them have to be assigned with different color. Of course, this coloring can be analogized to assign color of $G_1$. The edges $u_1v_1$ and $u_2v_2$ in $G_1$ have to be assigned with different color too and $G_1$ is class 1. This coloring then can be extended to assign color of $G$. So that, the suppose is false. $G$ is class 1.

Vertex $v$ misses color $\alpha$ means that no edges which incident with it be assigned by $\alpha$.

**Lemma 5.3.** Let $G$ be a petal graph such that $p(G) = 2$. Then $G$ is class 1.

**Proof.** It is proved by contradiction. Suppose that $G$ is class 2. It is critical. Let $P_w = v_1, w, v_2$ be a petal of $G$ whose size is 2 and $P_t = x, t, y$ be the other petal of $G$ (notice that its size is greater or equal from 2). Also, let $u_1, v_1, x, v_2, u_2$ be a path in $G_\Delta$. Let $G_1 = G - w$ (see Figure 4 for illustration). Notice that it is 3-colorable.

![Coloring of $G$](image1)

![Coloring of $G-w$](image2)

Figure 4. Illustration of proof of Lemma 5.3

There are two cases of coloring of $G_1$. First, vertices $v_1$ and $v_2$ miss different color. This case of coloring can be analogized immediately to assign color of $G$. Second, vertices $v_1$ and $v_2$ miss the same color. Let the coloring of $G_1$ be $\varphi_1(u_1v_1) : \alpha, \varphi_1(v_1x) : \beta, \varphi_1(v_2x) : \alpha, \varphi_1(v_2y) : \beta, \varphi_1(xt) : \alpha, \varphi_1(xt) : \gamma, \varphi_1(ty) : \beta$. This coloring of $G_1$ then can be analogized to assign color of $G$ by exchange $\varphi_1(xv_2)$ with $\varphi_1(xt)$ such that $G$ is 3-colorable. It means that $G$ is class 1, not class 2.

**Lemma 5.4.** Let $G$ be a petal graph such that $p(G) = \infty$. Then $G$ is class 1.

**Proof.** Again, this lemma is proved by contradiction. Suppose that $G$ is class 2 (it means that $G$ is 4-colorable). Let $K = v_0, v_2, \ldots, v_k, v_0$ be a cycle in $G_\Delta$ and the petals of $G$ are $P_{w_i} = v_i, w_i, y_i$ for $i = 0, 1, 2, \ldots, k$ and $j_i = w_iy_i$. Let $G_1$ and $H$ are two subgraph of $G$. The graph $G_1$ is defined as $G_1 = G - V(K)$. It is a graph contains the vertices in $G$ but not in $K$. The graph $H$ is defined as $H = G[E(K) \cup \bigcup_{i=1}^{k} E(P_{w_i})]$.

Applying Lemma 4.1, $G$ is critical so that $G_1$ is 3-colorable. Let the coloring of $G_1$ defined as $\varphi_1 : E(G_1) \rightarrow \{\alpha, \beta, \gamma\}$.

From the graph $H$, let $H^*$ is a graph obtained by splitting $v_0 \in V(H)$ into two vertices. The vertex that adjacent with $v_1$ is named by $z_1$ and the other one, that adjacent with $v_k$, is named by $z_k$. Let there is one-to-one correspondence between coloring of $H$ and $H^*$ where $z_1v_1$ and $z_kv_k$ are assigned by different color. The graph
$H^*$ is isomorphism with $L_{k+1}$. It is the graph defined in Lemma 4.3. Based on the lemma, there are proper 3-coloring of $H^*$ and satisfy $\varphi^*(z_1v_1) = \alpha_i$, $\varphi^*(f_i) = \varphi_1(f_i)$, for $i = 1, 2, \ldots k$ and $\varphi^*(z_kv_k) \neq \alpha$. This coloring can be analogized to assign colors of $H$. The coloring of $H^*, H$, $G_1$, and $G$ are illustrated on Figure 5.

The coloring of $G_1$ and $H$ can be extended to assign color of $G$ such that $G$ is 3-colorable. It is contradict with the suppose above. The graph $G$ is not class 2, but class 1.

By Lemma 5.2, 5.3, and 5.4, it is clear that all of petal graphs whose petal size are 1, 2 and infinite are class 1. For petal graphs whose petal size is $p(G) = p$ such that $3 \leq p < \infty$ is proved class 1 by Theorem 5.1 like Cariolaro and Cariolaro did it [2].

6. PROOF OF MAIN THEOREM

Like mentioned above, the term Main Theorem is refers to Theorem 5.1 that very useful on classifying petal graphs. The proof is below.

Proof. The proof is by contradiction, so suppose that $G$ is class 2. Let $v_0, v_0, v_1, v_2 \ldots v_p, u_p$ be a path whose length is $p + 2$ and contains the path $Y = v_0, v_1, v_2 \ldots v_p$ whose length is $p$. Both of the path are in $G_{\Delta}$. Let $P_{v_0} = v_0, u_0, v_p$ be a $p$-petal of $G$ and $P_{v_i} = v_i, w_i, y_i$, for $i = 1, 2, \ldots p - 1$, be the other petals containing $v_i$ and $f_i = w_iy_i$. Let $G_0 = G - u_0$. Since $G$ is class 2, it is critical and so that $G_0$ is class 1. It is illustrated in Figure 6 for $G$ is 4-petal.
Consider that there are two cases of coloring of $G_0$, which are defined as $\varphi_0 : E(G) \to \{\alpha, \beta, \gamma\}$. First case is the vertices $v_0$ and $v_p$ miss different color. This coloring can be analogized immediately to coloring of $G$. The second is the vertices $v_0$ and $v_p$ miss the same color. Now, it is necessary to prove that in this case the coloring of $G_0$ can be analogized too assign color of $G$.

Suppose that if 3-coloring of $G_0$ satisfy $\varphi_0(f_1) = \varphi_0(u_0v_0)$ or $\varphi_0(f_{p-1}) = \varphi_0(u_pv_p)$ then the vertices $v_0$ and $v_p$ miss the same color. By exchanging $\varphi_0(v_0v_1)$ with $\varphi_0(v_1w_1)$ or $\varphi_0(v_pv_{p-1})$ with $\varphi_0(v_{p-1}w_{p-1})$, now the vertices $v_0$ and $v_p$ miss different color (and, of course, it can be analogized to a 3-coloring of $G$). It is shown in Figure 1. It is a contradiction, but $G_0$ still 3-colorable. So that it is concluded that

$$\varphi_0(f_1) \neq \varphi_0(u_0v_0) \text{ and } \varphi_0(f_{p-1}) \neq \varphi_0(u_pv_p). \tag{6.1}$$

If there is no proper coloring so that the chromatic index of $G_0$ is 3 by exchanging $\varphi_0(v_0v_1)$ with $\varphi_0(v_1w_1)$ or $\varphi_0(v_pv_{p-1})$ with $\varphi_0(v_{p-1}w_{p-1})$, define $G_1 = G_0 - \{v_1, v_2, \ldots, v_{p-1}\}$ and consider that it is colored with the same way as coloring of $G_0$ which satisfy (6.1). Let $\varphi_1(u_0v_0) = \alpha$ and suppose that 3-coloring of $G_1$ satisfy $\varphi_1(u_pv_p) \neq \alpha$ (for example, $\beta$).

The graph $H$ is defined as $H = G_0[E(Y) \cup \bigcup_{i=1}^{p-1} E(P_{w_i})]$. Notice that $H \cong L_p$. It is the graph defined in Lemma 4.3. There are 3-colorings of $H$, which can be defined as $\hat{\varphi} : E(H) \to \{\alpha, \beta, \gamma\}$ such that $\hat{\varphi}(v_0v_1) = \gamma$, $\hat{\varphi}(f_i) = \varphi_1(f_i)$ for $1 \leq i \leq p-1$ and $\hat{\varphi}(v_pv_{p-1}) \neq \beta$.

The coloring of $G_0$ now can be defined as follow

$$\bar{\varphi}(e) = \begin{cases} 
\varphi_1(e) & \text{for } e \in E(G_1), \\
\hat{\varphi}(e) & \text{for } e \in E(H).
\end{cases} \tag{6.2}$$
Notice that this way of coloring, \( v_0 \) misses from color \( \beta \), but \( v_p \) does not. This contradiction let us conclude that

\[
\varphi_1(u_p v_p) = \varphi_1(u_0 v_0) = \alpha. \tag{6.3}
\]

Next, suppose that 3-coloring of \( G_1 \) satisfy \( \varphi_1(f_{p-2}) \neq \varphi_1(f_{p-1}) \). By (6.1) and (6.3) obtained \( \varphi_1(f_{p-1}) \neq \alpha \), for example \( \varphi_1(f_{p-1}) = \gamma \) and let \( \varphi_1(f_{p-2}) \neq \gamma \). Defined \( H_1 = H - \{v_p, w_{p-1}, y_{p-1}\} \). It is clear that \( H_1 \cong L_{p-1} \) so that \( H_1 \) is 3-colorable such that \( \hat{\varphi}_1(v_0 v_1) = \beta \), \( \hat{\varphi}_1(f_1) = \varphi_1(f_i) \) for \( i = 1, 2, \ldots p - 2 \) and \( \hat{\varphi}_1(v_{p-2} v_{p-1}) \neq \gamma \). If this coloring is extended to a coloring of \( H \) and by letting \( \hat{\varphi}(v_p v_{p-1}) = \gamma \), \( \hat{\varphi}(f_{p-1}) = \gamma \), and \( \hat{\varphi}(v_{p-1} w_{p-1}) \in \{\hat{\varphi}(v_{p-1} v_{p-2}), \gamma\} \) then the coloring of \( G_0 \) now can be defined the same as (6.2). This way of coloring, \( v_p \) misses from color \( \beta \) but \( v_0 \) does not. It is a contradiction. So, conclude that \( \varphi_1(f_{p-2}) = \varphi_1(f_{p-1}) \).

![Figure 7. The Coloring of \( G_1 \) which Satisfy 6.4 and The Coloring of \( H \)](image)

Next, it is concluded that for 3-coloring of \( G_1 \) satisfy

\[
\varphi_1(f_{p-i}) = \varphi_1(f_{p-1}) \text{ for all } i = 1, 2, \ldots p - 1. \tag{6.4}
\]

It is proved by induction.

1. Proved that it is true for \( i = 1 \).
   Equation (6.4) now become
   \[
   \varphi_1(f_{p-1}) = \varphi_1(f_{p-1}).
   \]

Of course, it is a true statement.

2. Assumed it is true for \( i = k \).
   Equation (6.4) become
   \[
   \varphi_1(f_{p-k}) = \varphi_1(f_{p-1}). \tag{6.5}
   \]

The equation (6.5) is true if the 3-coloring of \( G_0 \) can be defined as

\[
\hat{\varphi}(e) = \begin{cases} 
\varphi_1(e) & \text{for } e \in E(G_1), \\
\hat{\varphi}_k(e) & \text{for } e \in E(H_k),
\end{cases}
\]
where $H_k \cong L_k$. It was assumed that it is true.

(3) Proved it is true for $i = k + 1$.

Equation (6.4) become

$$\varphi_1(f_{p-(k+1)}) = \varphi_1(f_{p-1}).$$

Defined $H_{k+1} = H_k - \{v_{p-k}, w_{p-(k+1)}, y_{p-(k+1)}\}$. The graph $H_{k+1}$ is subgraph of $H_k$. By assigning color on $H_{k+1}$ in the same way as coloring on $H_k$, the coloring of $G_0$ next can be defined as

$$\bar{\varphi}(e) = \begin{cases} \varphi_1(e) & \text{for } e \in E(G_1), \\ \hat{\varphi}_{k+1}(e) & \text{for } e \in E(H_{k+1}), \end{cases}$$

and the rest of edges are colored by the same way as it was colored in $H_k$, so that it is concluded $\varphi_1(f_{p-(k+1)}) = \varphi_1(f_{p-1})$.

It was proved that $\varphi_1(f_{p-i}) = \varphi_1(f_{p-1})$ for all $i = 1, 2, \ldots p - 1$.

![Diagram](image)

Figure 8. Graph $G_1(\beta, \gamma)$ and $G_1(\beta, \gamma; f_2)$

Defined $G_1(\beta, \gamma)$. It is subgraph of $G_1$ which is induced by edges which are assigned by $\beta$ or $\gamma$. By interchanging $G_1(\beta, \gamma; f_2)$, the component of $G_1(\beta, \gamma)$ which contains $f_2$, the coloring of $G_1$ can be extended to coloring of $G_0$ and it can be extended again to coloring of $G$ so that $G$ is 3-colorable. Because $G$ is petal graph which $\Delta(G) = 3$, this is a contradiction. The graph $G$ is not class 2 but class 1.

\[\square\]

7. CONCLUSIONS AND REMARKS

From the explanation above, it is concluded that the petal size is the distance of basepoints in the core of graph. Its minimum value is defined as the petal size of the graphs. By proving one theorem, it shown how wonderful the petal graphs are. By single exception, it is all class 1. The exception is a graph which is obtained by deleting one vertex of Petersen graph which was known having mystery.

The readers who is interesting with this subject can study about the petal graphs whose maximum degree is 4 and minimum degree is 3 and its classification so that can simplify the problem of graphs classification.
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