ON THE METRIC DIMENSION OF LOLLIPPOP GRAPH, MONGOLIAN TENT GRAPH, AND GENERALIZED JAHANGIR GRAPH

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Abstract. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The distance $d(u,v)$ between vertex $u$ and $v$ in $G$ is the length of the shortest path from $u$ to $v$. A set of vertices in $S$ resolves a graph $G$ if every vertex is uniquely determined by its vector of distance to the vertices in $S$. The metric dimension of a graph $G$ is the minimum cardinality of a resolving set. In this paper we study the metric dimension of lollipop graph, Mongolian tent graph, and generalized Jahangir graph.

Keywords: metric dimension, resolving set, lollipop graph, Mongolian tent graph, generalized Jahangir graph

1. Introduction

The idea of resolving sets and minimum resolving sets has firstly appeared from Slater in 1975 also Harary and Melter in 1976. Slater [7] introduced the concept of a resolving set for a connected graph $G$ under the term locating set. He referred to a minimum resolving set as a reference set for $G$. Then the cardinality of a minimum resolving set is the location number of $G$. Harary and Melter [2] introduced these concepts using the term metric dimension.

Recently, many researchers have applied the concept of metric dimension to some of the graph classes. In 2000, Chartrand et al. [1] determined that the metric dimension of connected graph $G$ is 1 if and only if $G$ is a path graph $P_n$. Chartrand et al. [1] also determined that the metric dimension of complete graph $K_n$ is $n-1$. In 2007, the metric dimension of wheel graph and gear graph had been determined by Tomescu and Javaid [8]. They found that the metric dimension of wheel graph is $\left\lfloor \frac{2n+2}{3} \right\rfloor$ for $n \geq 7$ and the metric dimension of gear graph is $\left\lfloor \frac{2n}{3} \right\rfloor$ for $n \geq 4$. Then, in 2008 Iswadi et al. [4] determined the metric dimension of graphs with pendant edges. In this paper, we determine the metric dimension of lollipop graph, Mongolian tent graph, and generalized Jahangir graph.

2. Metric Dimension

Let $G$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. The distance between vertex $u$ and $v$ in $G$ denoted by $d(u,v)$, is the length of the shortest path from $u$ to $v$. Based on Chartrand et al. [1], for an ordered subset
W = \{w_1, w_2, \ldots, w_k\} of V(G), we define

\[ r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k)) \]

as the representation of v with respect to W. The set W is called a resolving set for G if \( r(u|W) \neq r(v|W) \) for every \( u, v \in V(G) \). A resolving set of minimum cardinality for a graph G is called a basis. Hence the number of vertices of a basis is called metric dimension of G, denoted by \( \text{dim}(G) \).

### 3. Metric Dimension of Lollipop Graph

Hoory [3] defined lollipop graph \( L_{m,n} \) for \( m \geq 3 \) as the graph obtained by joining a complete graph \( K_m \) to a path graph \( P_n \) with a bridge.

**Theorem 3.1.** Let \( L_{m,n} \) be a lollipop graph with \( m \geq 3 \). Then \( \text{dim}(L_{m,n}) = m-1 \).

**Proof.** We show that \( \text{dim}(L_{m,n}) \leq m-1 \) by choosing the resolving set \( W = \{v_1, v_2, v_3, \ldots, v_m-2, v_{m-1}\} \). The representations of each vertex with respect to W are

- \( r(v_1|W) = (0,1,1,\ldots,1,1) \);
- \( r(v_2|W) = (1,0,1,\ldots,1,1) \);
- \( r(v_3|W) = (1,1,0,\ldots,1,1) \);
- \( r(v_{m-1}|W) = (1,1,1,\ldots,1,0) \);
- \( r(v_m|W) = (1,1,1,\ldots,1,1) \).

All of those representations are distinct. Therefore \( \text{dim}(L_{m,n}) \leq m-1 \). Next we show that \( \text{dim}(L_{m,n}) \geq m-1 \). Suppose W is a basis of \( L_{m,n} \) with \( |W| < m-1 \). If every vertex of W belongs to \( \{v_k|1 \leq k \leq m\} \subset V(L_{m,n}) \), there are at least two vertices \( x \) and \( y \) in \( V(L_{m,n}) \) such that \( r(x|W) = r(y|W) = (1,1,\ldots,1) \), a contradiction. If some vertices of W belong to \( \{v_k|1 \leq k \leq m\} \subset V(L_{m,n}) \) and the other vertices belong to \( \{u_i|1 \leq i \leq n\} \subset V(L_{m,n}) \), then there are at least two vertices \( x \) and \( y \) in \( V(L_{m,n}) \) such that \( d(x,v_k) = d(y,v_k) = 1, \forall v_k \in W \) and \( d(x,u_i) = d(y,u_i) = i + 1, \forall u_i \in W \). This implies \( r(x|W) = r(y|W) \), a contradiction. If every vertex of W belongs to \( \{u_i|1 \leq i \leq n\} \subset V(L_{m,n}) \), then there are at least two vertices \( x \) and \( y \) in \( V(L_{m,n}) \) such that \( r(x|W) = r(y|W) = (2,3,4,\ldots) \), a contradiction. Hence, \( \text{dim}(L_{m,n}) = m-1 \).

### 4. Metric Dimension of Mongolian Tent Graph

Lee [5] defined Mongolian tent graph \( M_{m,n} \) as the graph obtained from the graph Cartesian product \( P_n \times P_n \) for odd n by adding an extra vertex above the graph and joining every other vertex of the top row to the additional vertex.
Theorem 4.1. Let $M_{m,n}$ be a Mongolian tent graph with $m \geq 2$ and $n \geq 3$. Then

$$dim(M_{m,n}) = \begin{cases} 3, & n = 3, 5; \\ \left\lfloor \frac{n}{2} \right\rfloor, & n = 7, 9; \\ \left\lfloor \frac{n}{2} \right\rfloor - 1, & n \geq 11. \end{cases}$$

Proof. There are three cases to prove the metric dimension of Mongolian tent graph.

Case 1. $n = 3, 5$.

(1) For $n = 3$. Let $W = \{v_1^1, v_2^1, v_3^1\}$ be a resolving set of the graph. The representations of each vertex with respect to $W$ are

$r(u|W) = (1, 2, 1)$;

$r(v_j^i|W) = \begin{cases} (i - 1, i, i + 1), & j = 1; \\ (i, i - 1, i), & j = 2; \\ (i + 1, i, i - 1), & j = 3. \end{cases}$

Next, we show that $M_{m,3}$ has no resolving set containing two elements by contradiction. Suppose that there is a resolving set $W$ containing two vertices. If $W = \{v_1^1, v_2^1\}$, then it follows $r(u|W) = (1, 2)$, a contradiction. Therefore $dim(M_{m,3}) = 3$.

(2) For $n = 5$. Let $W = \{v_1^1, v_2^1, v_3^1\}$ be a resolving set of the graph. The representations of each vertex with respect to $W$ are

$r(u|W) = (1, 1, 1)$;

$r(v_j^i|W) = \begin{cases} (i - 1, i + 1, i + 1), & j = 1; \\ (i, i, i + 2), & j = 2; \\ (i + 1, i - 1, i + 1), & j = 3; \\ (i + 2, i), & j = 4; \\ (i + 1, i + 1, i - 1), & j = 5. \end{cases}$

Next, we show that there is no resolving set containing two elements by contradiction. Suppose that there is a resolving set $W$ containing two vertices as follows. If $W = \{v_1^1, v_2^1\}$, then it follows $r(u|W) = (1, 1, 1)$, a contradiction. If $W = \{v_1^1, v_3^1\}$, then we have $r(v_1^3|W) = (1, 1, 1)$, a contradiction. Therefore $dim(M_{m,5}) = 3$.

Hence $dim(M_{m,n}) = 3$ for $n = 3, 5$. 
Case 2. \( n = 7,9 \).

(1) For \( n = 7 \). Let \( W = \{v_1^2, v_1^4, v_1^6\} \) be a resolving set of the graph. The representations of each vertex with respect to \( W \) are

\[
    r(u|W) = (2, 2, 2);
    \]

\[
    r(v_1^i|W) = \begin{cases} 
        (i, i + 2, i + 2), & j = 1; \\
        (i - 1, i + 1, i + 3), & j = 2; \\
        (i, i, i + 2), & j = 3; \\
        (i + 1, i - 1, i + 1), & j = 4; \\
        (i + 2, i, i), & j = 5; \\
        (i + 3, i + 1, i - 1), & j = 6; \\
        (i + 2, i + 2, i), & j = 7; \\
        (i + 3, i + 3, i + 1, i - 1), & j = 8; \\
        (i + 2, i + 2, i + 2, i), & j = 9. 
    \end{cases}
\]

Furthermore, we prove that there is no resolving set with two elements.
Assume that there is a resolving set \( W \) containing two vertices as follows.
If \( W = \{v_1^2, v_1^4, v_1^6\} \) and \( W = \{v_1^4, v_1^8\} \) then we have \( r(v_1^7|W) = r(v_1^8|W) \).
(2) For \( n = 9 \). Let \( W = \{v_1^2, v_1^4, v_1^6, v_1^8\} \) be a resolving set of the graph. The representations of each vertex with respect to \( W \) are

\[
    r(u|W) = (2, 2, 2, 2);
    \]

\[
    r(v_1^i|W) = \begin{cases} 
        (i, i + 2, i + 2, i + 2), & j = 1; \\
        (i - 1, i + 1, i + 3, i + 3), & j = 2; \\
        (i, i, i + 2, i + 2), & j = 3; \\
        (i + 1, i - 1, i + 1, i + 3), & j = 4; \\
        (i + 2, i, i, i + 2), & j = 5; \\
        (i + 3, i + 1, i - 1, i + 1), & j = 6; \\
        (i + 2, i + 2, i, i), & j = 7; \\
        (i + 3, i + 3, i + 1, i - 1), & j = 8; \\
        (i + 2, i + 2, i + 2, i), & j = 9. 
    \end{cases}
\]

Next, we show that there is no resolving set containing three elements by contradiction. Suppose that there \( M_{m,9} \) has a resolving set \( W \) with three vertices as follows. If \( W = \{v_1^2, v_1^4, v_1^6\} \), then we obtain \( r(v_1^8|W) = r(v_1^8|W) = (4, 4, 2), \) a contradiction. If \( W = \{v_1^2, v_1^4, v_1^8\} \), then it follows \( r(v_1^7|W) = r(v_1^8|W) = (3, 3, 1), \) a contradiction. If \( W = \{v_1^2, v_1^6, v_1^8\} \), then we have \( r(v_1^7|W) = r(v_1^8|W) = (1, 3, 3), \) a contradiction. While the existence of \( W = \{v_1^4, v_1^6, v_1^8\} \) would imply \( r(v_1^7|W) = r(v_1^8|W) = (2, 4, 4), \) a contradiction. Therefore \( dim(M_{m,9}) = 4 \).

Hence \( dim(M_{m,n}) = [\frac{n}{2}] \) for \( n = 7,9 \).
\[ \{v_1^2, v_1^4, \ldots, v_1^{\lceil \frac{n}{2} \rceil-1}, v_1^{\lceil \frac{n}{2} \rceil+1}, v_1^{\lceil \frac{n}{2} \rceil+2}, v_1^{\lceil \frac{n}{2} \rceil+4}, v_1^{n-1} \} \] for \( n \equiv 1 \pmod{4} \) so that every vertex has distinct representation with respect to \( W \). Thus, \( W \) is a resolving set and \( \dim(M_{m,n}) \leq \lfloor \frac{n}{2} \rfloor - 1 \) for \( n \geq 11 \). Now we show that \( \dim(M_{m,n}) \geq \lceil \frac{n}{2} \rceil - 1 \) for \( n \geq 11 \) by contradiction. Suppose that \( S = W - \{v_1^2\} \) is a resolving set. The cardinality of \( S \) is \( \lceil \frac{n}{2} \rceil - 2 \). In this case we have \( r(v_1^2|W) = r(v_2^3|W) \), a contradiction. This has also happened to all possibilities \( S \) which has cardinality of \( \lceil \frac{n}{2} \rceil - 2 \). Thus, the metric dimension of Mongolian tent graph for \( n \geq 11 \) is \( \lceil \frac{n}{2} \rceil - 1 \). This completes the proof of the theorem. 

5. Metric Dimension of Generalized Jahangir Graph

Mojdeh and Ghameshlou [6] defined generalized Jahangir graph \( J_{m,n} \) for \( n \geq 3 \) as a graph consisting of a cycle \( C_{mn} \) with one additional vertex which is adjacent to \( m \) vertices of \( C_{mn} \) at distance \( n \) to each other on \( C_{mn} \).

If \( B \) is a basis of \( J_{m,n} \), then it contains \( r \geq 2 \) vertices on \( C_{mn} \) (\( n \geq 6 \)) and we can order the vertices of \( B = \{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\} \) so that \( i_1 < i_2 < \ldots < i_r \). We shall say that the pairs of vertices \( \{v_{i_a}, v_{i_{a+1}}\} \) for \( 1 \leq a \leq r - 1 \) and \( \{v_{i_r}, v_{i_1}\} \) are pairs of neighboring vertices. According to Tomescu[8], we define the gap \( G_a \) for \( 1 \leq a \leq r - 1 \) as the set of vertices \( \{v_i|i_a < j < i_{a+1}\} \) and \( G_r = \{v_i|1 \leq j < i_r \text{ or } i_r < j < mn\} \). Thus we have \( r \) gaps, some of which may be empty. The gap \( G_a \) and \( G_b \) are neighboring gaps when \( |a - b| = 1 \text{ or } r - 1 \).

The following lemmas are used to determine the metric dimension of generalized Jahangir graph \( J_{m,n} \) with \( m \geq 3 \) and \( n \geq 6 \).

**Lemma 5.1.** The central vertex \( v \) of \( J_{m,n} \) does not belong to any basis of \( J_{m,n} \).

**Proof.** We prove the result by contradiction. Let \( B \) be a basis of \( J_{m,n} \) that contains \( v \). Since \( B \setminus \{v\} \) is not a basis, there exist vertices \( u \) and \( u' \) such that \( d(u, x) = d(u', x) \) for every \( x \in B \setminus \{v\} \). Clearly \( B = \{v\} \) is not a basis, so \( B \setminus \{v\} \neq \emptyset \). If neither \( u = v \) nor \( u' = v \), then \( d(u, v) = d(u', v) \) and \( B \) is not a basis of \( J_{m,n} \). We can assume that \( u' = v \), and without loss of generality \( u \) is the vertex \( v_3^1 \). In this case \( d(v_3^1, x) = d(v_3^1, x) \) for any \( x \in B \), a contradiction.

**Lemma 5.2.** Let \( B \) be a basis of \( J_{m,n} \), then \( B \) contains at most \( 8 \) vertices for \( m = 3 \) and \( (2m - 1) \) vertices otherwise.

**Proof.** Suppose that there is a gap of \( B \) containing \( 9 \) consecutive vertices \( x_1, \ldots, x_9 \) of \( C_{3n} \). In this case \( r(x_1|B) = r(x_5|B) \), a contradiction. If there is a gap of \( B \) on \( C_{mn} \) (\( m > 3 \)) containing \( 2m \) consecutive vertices \( x_1, x_2, \ldots, x_{2m} \) then it follows that \( r(x_{m-1}|B) = r(x_{m+1}|B) \), a contradiction. 

The gaps of \( B \) containing \( 8 \) vertices for \( m = 3 \) and \( (2m - 1) \) vertices otherwise are called major gaps, while the others are called minor gaps.
Lemma 5.3. Let \( B \) be a basis of \( J_{m,n} \), then \( B \) contains at most one major gap for \( m = 3 \) and \( \left\lceil \frac{n}{2} \right\rceil \) major gaps for any even integer \( m \).

Proof. For \( m = 3 \), suppose that there are two distinct major gaps \( G_1 = \{x_1, \ldots, x_8\} \) and \( G_2 = \{y_1, \ldots, y_8\} \). In this case \( r(x_3|B) = r(y_3|B) \), a contradiction. For any even integer \( m \), suppose that there are \( \left\lceil \frac{n}{2} \right\rceil + 1 \) distinct major gaps \( G_1, \ldots, G_{\left\lceil \frac{n}{2} \right\rceil + 1} \) with \( G_1 = \{x_1, \ldots, x_{2^{m-1}}\} \), \( G_2 = \{x_2, \ldots, x_{2^{m-1}}\} \), and so on. In this case, \( r(x_2^m|B) = r(x_2^m|B) = \ldots = r(x_{\left\lceil \frac{n}{2} \right\rceil + 1}^m|B) \), a contradiction. \( \Box \)

Lemma 5.4. Let \( B \) be a basis of \( J_{m,n} \) for any odd integer \( m \) \((m > 3)\), then \( B \) contains \( \left\lceil \frac{n}{2} \right\rceil \) major gaps and at most one minor gap.

Proof. It is not difficult to show that \( J_{m,n} \) has at most \( \left\lceil \frac{n}{2} \right\rceil \) major gaps. Now, we show that \( J_{m,n} \) has at least \( \left\lceil \frac{n}{2} \right\rceil \)-1 distinct major gaps \( G_1, \ldots, G_{\left\lceil \frac{n}{2} \right\rceil - 1} \) with \( G_1 = \{x_1, \ldots, x_{2^{m-1}}\} \), \( G_2 = \{x_2, \ldots, x_{2^{m-1}}\} \), and so on. In this case \( r(x_1^m|B) = r(x_2^m|B) = \ldots = r(x_{\left\lceil \frac{n}{2} \right\rceil - 1}^m|B) \), a contradiction. Next, we show that \( J_{m,n} \) contains at most one minor gap. By Lemma 5.2, major gaps of this graph contains \( (2m - 1) \) vertices. It follows that \( J_{m,n} \) has no minor gap when \( n \) is even. When \( n \) is odd, let \( G_i = \{x_1, \ldots, x_{2^{m-1}}\} \), \( i = 1, \ldots, r - 2 \) be major gaps. The existence of two minor gaps \( G_{r-1} \) and \( G_r \) would imply \( r(x_1^m|B) = r(x_2^m|B) = \ldots = r(x_r^m|B) \), a contradiction. \( \Box \)

Lemma 5.5. Let \( B \) be a basis of \( J_{m,n} \), then any two neighboring gaps, one of which being a major gap, contain together at most 13 vertices for \( m = 3 \) and \( (3m - 2) \) vertices otherwise.

Proof. There are three cases to consider according to the values of \( m \).

Case 1. \( m = 3 \). If the major gap \( G_1 = \{x_1, \ldots, x_8\} \) has a neighboring minor gap \( G_2 = \{y_1, \ldots, y_8\} \), it causes \( r(x_6|B) = r(y_3|B) \) which becomes a contradiction.

Case 2. \( m \) is even. If the major gap \( G_1 = \{x_1, \ldots, x_{2^{m-1}}\} \) has a neighboring minor gap \( G_2 = \{y_1, \ldots, y_m\} \), it causes \( r(x_{2^{m-1}}|B) = r(y_1|B) \) which becomes a contradiction.

Case 3. \( m \) is odd. By Lemma 5.2 and 5.4 all of major gaps contain together \( (2m - 1)\left\lceil \frac{n}{2} \right\rceil \) vertices and \( B \) consists of \( \left\lceil \frac{n}{2} \right\rceil \) vertices. It means, for any odd \( n \), there are \( m \) vertices which do not belong to any gap or basis yet. Since \( J_{m,n} \) with odd \( m \) only has one minor gap, the gap must contain \((m - 1)\) vertices. Therefore, the major gap and the minor gap together contain \((2m - 1) + (m - 1) = (3m - 1)\) vertices. \( \Box \)

Lemma 5.6. Let \( B \) be a basis of \( J_{m,n} \), then any two minor neighboring gaps contain together at most 10 vertices for \( m = 3 \) and \( (2m - 2) \) vertices for any even integer \( m \).
Proof. By Lemma 5.5, for $m = 3$ the major gap can have a neighboring minor gap containing at most 5 vertices. So any two minor neighboring gaps contain together at most 10 vertices. If $m$ is even, the minor gaps which contain more than $(m - 1)$ vertices do not exist. So any two minor gaps contain together at most $2m - 2$ vertices.

\[ \text{dim}(J_{m,n}) = \begin{cases} 
\left\lfloor \frac{n}{2} \right\rfloor, & \text{if } m = 3; \\
\left\lfloor \frac{2n+2}{3} \right\rfloor, & \text{if } m \text{ is even; } \\
\left\lceil \frac{n}{2} \right\rceil, & \text{if } m \text{ is odd.}
\end{cases} \]

Proof. By Lemma 5.1 we have seen that the central vertex $v$ of $J_{m,n}$ does not belong to any basis $B$ of $J_{m,n}$. There are three cases to prove the theorem.

1. Case 1. $m = 3$.
Suppose that $W = \{v_1, v_3, \ldots \}$ is a resolving set with $\left\lceil \frac{n}{2} \right\rceil$ vertices. The graph $J_{3,n}$ contains at most one major gap and all other gaps are minor gaps with 5 vertices. Note that $r(v|W) = (2, 2, \ldots, 2)$ and the representations of each vertex $x \in J_{3,n}$, $x \neq v$, with respect to $W$ is different from $r(v|W)$. Therefore, $\text{dim}(J_{3,n}) \leq \left\lceil \frac{n}{2} \right\rceil$. Now we show that $\text{dim}(J_{3,n}) \geq \left\lfloor \frac{n}{2} \right\rfloor$. Let $B$ be a basis of $J_{3,n}$ and $|B| = r$. This implies there are $r$ gaps on $C_{3n}$ denoted by $G_1, \ldots, G_r$. By Lemma 5.3 at most one of the gaps, say $G_1$, is a major gap. By Lemmas 5.5 and 5.6, we can write $|G_1| + |G_2| \leq 13$, $|G_r| + |G_1| \leq 13$, and $|G_i| + |G_{i+1}| \leq 10$ for $i = 2, \ldots, r - 1$. By summing these inequalities, we have

\[ 2(3n - r) = 2 \sum_{i=1}^{r} |G_i| \leq 10r + 6; \]

so $r \geq (n - 1)/2$. Since $r$ is integer, for each $n \equiv 0, 1(\mod 2)$ we obtain $r \geq \left\lfloor \frac{n}{2} \right\rfloor$.

2. Case 2. $m$ is even.
Suppose that $W = \{v_1, v_2, v_4, v_5, \ldots \}$ is a resolving set with $\left\lfloor \frac{2n+2}{3} \right\rfloor$ vertices. This graph contains at most $\left\lfloor \frac{n}{3} \right\rfloor$ major gap and all other gaps are minor gaps with $(m - 1)$ vertices. Note that $r(v|W) = (\frac{m}{2} + 1, \frac{m}{2} + 1, \ldots, \frac{m}{2} + 1)$ and the representations of each vertex $x \in J_{m,n}$, $x \neq v$, with respect to $W$ is different from $r(v|W)$. Therefore, $\text{dim}(J_{m,n}) \leq \left\lfloor \frac{2n+2}{3} \right\rfloor$. Now we show that $\text{dim}(J_{m,n}) \geq \left\lceil \frac{2n+2}{3} \right\rceil$. Let $B$ be a basis of $J_{m,n}$ and $|B| = r$. This implies there are $r$ gaps on $C_{mn}$ denoted by $G_1, \ldots, G_r$. Again, by Lemmas 5.3, 5.5 and 5.6, we have

\[ 2(mn - r) = 2 \sum_{i=1}^{r} |G_i| \leq 2(3m - 2)\left\lceil \frac{n}{3} \right\rceil + (2m - 2)(r - 2\left\lfloor \frac{n}{3} \right\rfloor); \]

\[ 2(mn - r) = 2 \sum_{i=1}^{r} |G_i| \leq 2(3m - 2)\left\lfloor \frac{n}{3} \right\rceil + (2m - 2)(r - 2\left\lfloor \frac{n}{3} \right\rfloor); \]

\[ 2(mn - r) = 2 \sum_{i=1}^{r} |G_i| \leq 2(3m - 2)\left\lfloor \frac{n}{3} \right\rceil + (2m - 2)(r - 2\left\lfloor \frac{n}{3} \right\rfloor); \]

\[ 2(mn - r) = 2 \sum_{i=1}^{r} |G_i| \leq 2(3m - 2)\left\lfloor \frac{n}{3} \right\rceil + (2m - 2)(r - 2\left\lfloor \frac{n}{3} \right\rfloor); \]
so \( r \geq n - \lfloor \frac{n}{3} \rfloor \). Since \( r \) is integer, for each \( n \equiv 0, 1, 2(\mod 3) \) we obtain 
\( r \geq \lfloor \frac{2n+2}{3} \rfloor \).

(3) Case 3. \( m \) is odd.

Suppose that \( W = \{v_1^{\lfloor \frac{n}{2} \rfloor}, v_3^{\lfloor \frac{n}{2} \rfloor}, v_5^{\lfloor \frac{n}{2} \rfloor}, \ldots \} \) is a resolving set with \( \lfloor \frac{n}{2} \rfloor \) vertices. This graph contains at most \( \lfloor \frac{n}{2} \rfloor \) major gap and all other gaps are minor gaps with \( (m-1) \) vertices. Note that \( r(v|W) = (\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor, \ldots, \lfloor \frac{n}{2} \rfloor) \) and the representations of each vertex \( x \in J_{m,n} \), \( x \neq v \), with respect to \( W \) is different from \( r(v|W) \). Therefore, \( dim(J_{m,n}) \leq \lfloor \frac{n}{2} \rfloor \). Now we show that \( dim(J_{m,n}) \geq \lceil \frac{n}{2} \rceil \). Let \( B \) be a basis of \( J_{m,n} \) and \( |B| = r \). This implies there are \( r \) gaps on \( C_{mn} \) denoted by \( G_1, \ldots, G_r \). By Lemma 5.4 at most one of them, say \( G_1 \), is a major gap. By Lemmas 5.2 and 5.5, we can write 
\[
\begin{align*}
|G_1| + |G_2| & \leq 3m - 2, \\
|G_r| + |G_1| & \leq 3m - 2, \\
|G_i| + |G_{i+1}| & \leq 4m - 2
\end{align*}
\]
for \( i = 2, \ldots, r - 1 \). By summing these inequalities, we have 
\[
2(mn - r) = 2 \sum_{i=1}^{r} |G_i| \leq 2(3m - 2) + (r-2)(4m - 2);
\]
so \( r \geq \lceil \frac{n}{2} \rceil \).

Thus completes the proof of the theorem. \( \square \)

6. CONCLUSION

We have found the metric dimension of lollipop graph, Mongolian tent graph, and generalized Jahangir graph as stated in Theorem 3.1, 4.1, and 5.7, respectively.

References


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