ON THE PARTITION DIMENSION OF A LOLLIPOP GRAPH, A GENERALIZED JAHANGIR GRAPH, AND A $C_n *_2 K_m$ GRAPH

Maylinda Purna Kartika Dewi and Tri Atmojo Kusmayadi
Department of Mathematics
Faculty of Mathematics and Natural Sciences
Sebelas Maret University

Abstract. Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_n\}$. The vertex set $V(G)$ is divided into some partitions, which are $S_1, S_2, \ldots, S_k$. For every vertex $v \in V(G)$ and an ordered k-partition $\Pi = \{S_1, S_2, \ldots, S_k\}$, the representation of v with respect to Π is $r(v|\Pi) = (d(v, S_1), d(v, S_2), \ldots, d(v, S_k))$, where $d(v, S_i)$ represents the distance of the vertex v to each partition in Π. The set Π is said to be resolved partition of G if the representation $r(v|\Pi)$ are distinct, for every vertex v in G. The minimum cardinality of resolving k-partition of $V(G)$ is called the partition dimension of G, denoted by $pd(G)$. In this research, we determine the partition dimension of a lollipop graph $L_{m,n}$, a generalized Jahangir graph $J_{m,n}$, and a $C_n *_2 K_m$ graph.
Keywords : *partition dimension, resolving partition, lollipop graph, generalized Jahangir graph, $C_n *_2 K_m$ graph*

1. Introduction

The partition dimension of a graph was introduced by Chartrand et al. [2] in 1998. There is another concept of metric dimension form. We assume that G is a graph with a vertex set $V(G)$, such that $V(G)$ can be divided into any partition set S. The set Π with $S \in \Pi$ is a resolving partition of G if for each vertex in G has distinct representation with respect to Π, and Π is an ordered $k-$partition. The minimum cardinality of resolving $k-$partitions of $V(G)$ is called a partition dimension of G, denoted by $pd(G)$.

Many researchers have conducted research in determining the partition dimension for specific graph classes. Tomescu et al. [9] in 2007 found the partition dimension of a wheel graph. Javaid and Shokat [6] in 2008 determined the partition dimension of some wheel related graphs, such as a gear graph, a helm graph, a sunflower graph, and a friendship graph. Asmiati [1] in 2012 investigated the partition dimension of a star amalgamation. Hidayat [5] in 2015 determined the partition dimension of some graph classes, they are a $(n, t) - kite$ graph, a barbell graph, a double cones graph, and a $K_1 + (P_m \odot K_n)$ graph. Danisa [4] in 2015 found the partition dimension of a flower graph, a $3-$fold wheel graph, and a $(K_m \times P_n) \odot K_1$. Puspitaningrum [8] in 2015 also found the partition dimension of a closed helm graph, a $W_n \times P_m$ graph, and a $C_m \odot K_{1,n}$ graph. In this research, we determine the partition dimension of a lollipop graph, a generalized Jahangir graph, and a $C_n *_2 K_m$ graph.

2. Partition Dimension

The following definition and lemma were given by Chartrand et al. [3]. Let G be a connected graph. For a subset S of $V(G)$ and a vertex v of G, the distance $d(v, S)$ between v and S is defined as $d(v, S) = min\{d(v, x)|x \in S\}$. For an ordered k-partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of $V(G)$ and a vertex v of G, the representation of v with respect to Π is defined as the k-vector $r(v|\Pi) = (d(v, S_1), d(v, S_2), \ldots, d(v, S_k))$. The partition Π is called a resolving partition if the k-vectors $r(v|\Pi), v \in V(G)$ are distinct. The minimum k for which there is a resolving k-partition of $V(G)$ is called the partition dimension of G, denoted by $pd(G)$.

Lemma 2.1. *Let G be a connected graph, then*

(1) $pd(G) = 2$ *if and only if* $G = P_n$ *for* $n \geq 2$,
(2) $pd(G) = n$ *if and only if* $G = K_n$, *and*
(3) $pd(G) = 3$ *if* $G = n$-cycle *for* $n \geq 3$.

Lemma 2.2. *Let Π be a resolving partition of $V(G)$ and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for all $w \in V(G) - \{u, v\}$, then u and v belong to distinct elements of Π.*

Proof. Let $\Pi = \{S_1, S_2, \ldots, S_k\}$, where u and v belong to the same element, say S_i, of Π. Then $d(u, S_i) = d(v, S_i) = 0$. Since $d(u, w) = d(v, w)$ for all $w \in V(G) - \{u, v\}$, we also have that $d(u, S_j) = d(v, S_j)$ for all j, where $1 \leq j \neq i \leq k$. Therefore, $r(u|\Pi) = r(v|\Pi)$ and Π is not a resolving partition. □

3. The Partition Dimension of a Lollipop Graph

Weisstein [10] defined the lollipop graph $L_{m,n}$ for $m \geq 3$ as a graph obtained by joining a complete graph K_m to a path P_n with a bridge.

Theorem 3.1. *Let $L_{m,n}$ be a lollipop graph with $m \geq 3$ and $n \geq 1$, then*

$$pd(L_{m,n}) = m.$$

Proof. Let $V(L_{m,n}) = V(K_m) \cup V(P_n)$, where $V(K_m) = \{v_1, v_2, \ldots, v_{m-1}, v_m\}$ and $V(P_n) = \{u_1, u_2, \ldots, u_{n-1}, u_n\}$. Take a vertex $v \in V(K_m)$ such that v is adjacent to $u \in V(P_n)$. Based on Lemma 2.1.1 and Lemma 2.1.2, the partition dimension of $L_{m,n}$ is $pd(L_{m,n}) \geq m$. Next, we show that the partition dimension of $L_{m,n}$ is $pd(L_{m,n}) = m$. For each $v, u \in V(L_{m,n})$, let $\Pi = \{S_1, S_2, \ldots, S_m\}$ be the resolving partition where $S_i = \{v_i\}$ for $1 \leq i \leq m-1$ and $S_m = \{v_m, u_1, \ldots, u_n\}$. We obtain the representation between all vertices of $V(L_{m,n})$ with respect to Π as follows $r(v_m|\Pi) = (d(v_1, S_i), \ldots, d(v_m, S_i))$ and $r(u_n|\Pi) = (n, n+1, n+1, \ldots, n+1, 0)$ where

$$d(v_m, S_i) = \begin{cases} 0, & \text{for } S_i \in \Pi \text{ with } i = m, \\ 1, & \text{for } S_i \in \Pi \text{ with } i \neq m. \end{cases}$$

Therefore, for each vertex $v \in V(L_{m,n})$ has a distinct representation with respect to Π. Now, we show that the cardinality of Π is m. The cardinality of Π is the number of $k-$partition in S_i and S_m for $1 \leq i \leq m-1$. The number of $k-$partition in S_i for $1 \leq i \leq m-1$, is $m-1$. The number of $k-$partition in S_m is 1, so we have $|\ \Pi\ | = (m-1) + 1 = m$. Furthermore, $\Pi = \{S_1, S_2, \ldots, S_m\}$ is a resolving partition of $L_{m,n}$ with m elements. Hence, the partition dimension of lollipop graph is $pd(L_{m,n}) = m$. □

4. The Partition Dimension of a Generalized Jahangir Graph

Mojdeh and Ghameshlou [7] defined the generalized Jahangir graph $J_{m,n}$ with $n \geq 3$ as a graph on $nm+1$ vertices consisting of a cycle C_{mn} and one additional vertex which is adjacent to n vertices of C_{mn} at m distance to each other on C_{mn}. Let $V(J_{m,n}) = V(C_{mn}) \cup \{u_1\}$ or $V(J_{m,n}) = V(P_m) \cup V(W_n)$ where $V(P_m) = \{v_1, v_2, \ldots, v_{n(m-1)}\}$ and $V(W_n) = \{u_1, u_2, \ldots, u_{n+1}\}$, we have $V(J_{m,n}) = \{u_1, u_2, \ldots, u_{n+1}, v_1, v_2, \ldots, v_{n(m-1)}\}$ with $m, n \in \mathbb{N}$.

Lemma 4.1. *Let G be a subgraph of a generalized Jahangir graph $J_{m,n}$ for $m \geq 3$, $n \geq 3$ and Π is a resolving partition of G. The distance of vertex u_1 to the other vertices are $d(u_1, u_i) = 1$ and $d(u_1, v_k) = 2$ for $1 < i \leq n+1$ and $1 \leq k \leq n(m-1)$. The distance of vertex u to the other vertex u are same, $d(u_i, u_j) = 2$ for $1 < i \leq n+1$, $1 < j \leq n+1$ and $i \neq j$.*

Proof. Let u_1 be a central vertex of $V(G) = V(J_{m,n})$. Since for each vertex u_i is connected with central vertex, so $d(u_1, u_i) = 1$ and $d(u_i, u_j) = 2$ with $1 < i \leq n+1$, $1 < j \leq n+1$, and $i \neq j$. If vertex v_k is vertex of path P_m and has minimum distance of vertex u_i, then $d(u_1, v_k) = 2$ for $1 \leq k \leq n(m-1)$. □

Theorem 4.1. *Let $J_{m,n}$ be a generalized Jahangir graph for $m \geq 3$ and $n \geq 3$ then*

$$pd(J_{m,n}) = \begin{cases} 3, & \text{for } n = 3, 4, 5; \\ \lfloor \frac{n}{2} \rfloor + 1, & \text{for } n \geq 6. \end{cases}$$

Proof. Let $J_{m,n}$ be a generalized Jahangir graph for $m \geq 3$, $n \geq 3$ and a vertex set $V(J_{m,n})$. The proof can be divided into two cases according to the values of n.

(1) Case $n = 3, 4, 5$

(a) For $n = 3$

Let $a = \{v_1, \ldots, v_m\}$, $b = \{v_1, \ldots, v_{m-1}\}$, $c = \{v_{m+1}, \ldots, v_{n(m-1)}\}$, and $d = \{v_m, \ldots, v_{2(m-1)}\}$, then

$$S_1 = \{u_1, \ldots, u_{n+1}, v_{2m-1}, \ldots, v_{mn-3}\},$$

$$S_2 = \begin{cases} a, & \text{for } m = 3; \\ b, & \text{for } m \geq 4; \end{cases} \quad S_3 = \begin{cases} c, & \text{for } m = 3; \\ d, & \text{for } m \geq 4. \end{cases}$$

(b) For $n = 4$

Let $a = \{u_1, \ldots, u_{n+2}, v_{(n(m-1))-1}\}$,
$b = \{u_1, \ldots, u_{n+2}, v_{3m-2}, \ldots, v_{n(m-1)}\}$,
$c = \{u_1, \ldots, u_{n+2}, v_{2m-1}, v_{3m-2}, \ldots, v_{n(m-1)}\}$,
$d = \{v_1, \ldots, v_m\}$,
$e = \{v_1, \ldots, v_{m-1}, v_{2(m+1)}, v_{m(n-1)-3}\}$,
$f = \{v_1, \ldots, v_{m-1}, v_{m(n-1)-3}\}$,
$g = \{v_{m+1}, \ldots, v_{n(m-1)}\}$,
$h = \{v_m, \ldots, v_{2(m-1)}, v_{2m-1}, \ldots, v_{(m-1)(n-1)-1}\}$,
$i = \{v_m, \ldots, v_{2(m-1)}, v_{2m}, \ldots, v_{(m-1)(n-1)-1}\}$, and
$j = \{v_m, \ldots, v_{2(m-1)}, v_{2m}, v_{2m+1}, v_{2m+3}, \ldots, v_{(m-1)(n-1)-1}\}$ then

$$S_1 = \begin{cases} a, & \text{for } m = 3; \\ b, & \text{for } m = 5; \\ c, & \text{for } m \geq 4, m \neq 5; \end{cases} \quad S_2 = \begin{cases} d, & \text{for } m = 3; \\ e, & \text{for } m = 9; \\ f, & \text{for } m \geq 4, m \neq 9; \end{cases}$$

$$S_3 = \begin{cases} g, & \text{for } m = 3; \\ h, & \text{for } m = 5; \\ i, & \text{for } m \geq 4, m \neq 5, 9; \\ j, & \text{for } m = 9. \end{cases}$$

For $n = 5$ is same as above. The representations of $r(u|\Pi)$ and $r(v|\Pi)$, with $u, v \in V(J_{m,n})$ are

$r(u_1|\Pi) = (0, 2, 2)$, $r(u_2|\Pi) = (0, 1, 3)$, $r(u_3|\Pi) = (0, 1, 1)$,
$r(u_n|\Pi) = (0, 3, 1)$, $r(v_1|\Pi) = (1, 0, 4)$, $r(v_2|\Pi) = (2, 0, 5)$,
$\ldots$, $r(v_{n(m-1)-1}|\Pi) = (0, 3, 5)$, $r(v_{n(m-1)}|\Pi) = (0, 2, 4)$.

So the representation of $r(u|\Pi)$ and $r(v|\Pi)$ are distinct and we conclude that Π is a resolving partition of $J_{m,n}$ with 3 elements. Furthermore, we prove that there is no resolving partition with 2 elements. Suppose that $J_{m,n}$ has a resolving partition with 2 elements, $\Pi = \{S_1, S_2\}$. Based on Lemma 4.1, it is a contradiction. Therefore, $J_{m,n}$ has no a resolving partition with 2 elements. Thus, the partition dimension of $J_{m,n}$ is $pd(J_{m,n}) = 3$ for $n = 3, 4, 5$.

(2) Case $n \geq 6$.

Let $\Pi = \{S_1, S_i\}$ be a resolving partition of $J_{m,n}$ where $S_1 = \{u_i, v_j\}$, for $1 \leq i \leq n+1$, $1 \leq j \leq n(m-1)$ and $S_i = \{v_k\}$, for $1 < i \leq \lfloor \frac{n}{2} \rfloor + 1, 1 \leq k \leq n(m-1), k \neq j$. The representation for each vertex on $J_{m,n}$ with respect to Π as follows

$$
\begin{array}{ll}
r(u_1|\Pi) = (0, 2, 2, \ldots, 2), & r(u_2|\Pi) = (0, 1, 3, \ldots, 3), \\
r(u_{n+1}|\Pi) = (0, 1, 1, \ldots, 3), & \ldots, \\
r(v_1|\Pi) = (1, 0, 4, \ldots, 4), & r(v_2|\Pi) = (2, 0, \ldots, 5), \\
\ldots, & r(v_{(m-1)}|\Pi) = (1, 0, 2, \ldots, 4), \\
r(v_m|\Pi) = (1, 2, 0, \ldots, 4), & r(v_{m+1}|\Pi) = (2, 3, 0, \ldots, 5), \\
\ldots, & r(v_{2(m-1)}|\Pi) = (1, 4, 2, 0, \ldots), \\
r(v_{2m}|\Pi) = (2, 5, 3, 0, \ldots), & \ldots, \\
r(v_{n(m-1)-1}|\Pi) = (0, 2, 5, \ldots, 5), & r(v_{n(m-1)}|\Pi) = (0, 1, 4, \ldots, 4).
\end{array}
$$

Since for each vertex of $J_{m,n}$ has a distinct representation with respect to Π, so Π is a resolving partition of $J_{m,n}$ with $\lfloor \frac{n}{2} \rfloor + 1$ elements.

Next, we show that the cardinality of Π is $\lfloor \frac{n}{2} \rfloor + 1$. The cardinality of Π is the number of $k-$partition in S_1 and S_i for $1 < i \leq \lfloor \frac{n}{2} \rfloor$. The number of $k-$partition in S_1 is 1. The number of $k-$partition in S_i for $1 < i \leq \lfloor \frac{n}{2} \rfloor$ is $\lfloor \frac{n}{2} \rfloor$. We obtain $\mid \Pi \mid = 1 + \lfloor \frac{n}{2} \rfloor = \lfloor \frac{n}{2} \rfloor + 1$. Therefore, $\Pi = \{S_1, S_2, \ldots, S_{\lfloor \frac{n}{2} \rfloor + 1}\}$ is a resolving partition of $J_{m,n}$.

We show that $J_{m,n}$ has no $pd(J_{m,n}) < \lfloor \frac{n}{2} \rfloor + 1$. We may assume that Π is a resolving partition of $J_{m,n}$ where $pd(J_{m,n}) < \lfloor \frac{n}{2} \rfloor + 1$. So, for each vertex $v \in V(J_{m,n})$ has a distinct representation $r(v_i|\Pi)$ and $r(v_j|\Pi)$. If we select $\Pi = \{S_1, S_2, S_3, \ldots, S_{\lfloor \frac{n}{2} \rfloor}\}$ as a resolving partition then there is a partition class containing any 2 vertices of v_j. By using Lemma 4.1, the distance of vertex u_1 to the others have the same distance, and $r(v_i|\Pi) = r(v_j|\Pi)$. It is a contradiction. Hence, the partition dimension of $J_{m,n}$ is $pd(J_{m,n}) = \lfloor \frac{n}{2} \rfloor + 1$ for $n \geq 6$.

□

5. The Partition Dimension of a $C_n *_2 K_m$ Graph

$C_n *_2 K_m$ graph with $n \geq 3$ and $m \geq 2$ is a graph obtained from edge amalgamation, which combine an edge of C_n and an edge of K_m to be one incident edge with vertex x and y. Vertex x and y also as owned by C_n and K_m. Let $V(C_n *_2 K_m) = A \cup B \cup \{x\} \cup \{y\}$ with $A \in V(C_n) \backslash \{u_1, u_2\}$ and $B \in V(K_m) \backslash \{v_1, v_2\}$.

A vertex set $A = \{u_3, u_4, \ldots, u_n\}$ and $B = \{v_3, v_4, \ldots, v_m\}$, vertex $x = u_1 * v_1$ and $y = u_2 * v_2$, so $V(C_n *_2 K_m) = \{x, y, u_3, u_4, \ldots, u_n, v_3, v_4, \ldots, v_m\}$.

Lemma 5.1. *Let G be a subgraph of $C_n *_2 K_m$ graph for $n \geq 3$, $m \geq 2$ and Π is a resolving partition of G. Then the distance of vertex x and y to vertex v_i are $d(x, v_i) = d(y, v_i) = 1$ for $3 \leq i \leq m$. Furthermore, x and y belong to distinct elements of Π.*

Proof. Let $\Pi = \{S_1, S_2, \ldots, S_m\}$ be a resolving partition of G and $x, y, v_i \in G$. Since vertex $x, y, v_i \in V(K_m)$ with $3 \leq i \leq m$, so the distance of vertex x and y to vertex v_i are $d(x, v_i) = d(y, v_i) = 1$ for $3 \leq i \leq m$. Based on Lemma 2.2 then vertex x and y belong to distinct elements of Π. □

Theorem 5.1. *Let $C_n *_2 K_m$ be a graph of resulting an edge amalgamation of C_n and K_m for $n \geq 3$ and $m \geq 2$ then*

$$pd(C_n *_2 K_m) = \begin{cases} 3, & \text{for } m = 2, 3, 4; \\ m - 1, & \text{for } m \geq 5. \end{cases}$$

Proof. Let $C_n *_2 K_m$ be a graph of resulting an edge amalgamation for $n \geq 3$ and $m \geq 2$. A vertex set $V(C_n *_2 K_m)$ where (x, y) are joining edge among edge (u_1, u_2) and (v_1, v_2). The proof can be divided into two cases based on the values of m.

(1) Case $m = 2, 3, 4$.

According to Lemma 2.1, the partition dimensions of $C_n *_2 K_m$ are $pd(C_n *_2 K_m) \neq 2$ and $pd(C_n *_2 K_m) \geq 3$.

(a) We prove that $pd(C_n *_2 K_m) \neq 2$. Based on Lemma 2.1.1, $pd(G) = 2$ if and only if $G = P_n$. Since $C_n *_2 K_m$ cannot be formed be a path, then $pd(C_n *_2 K_m) \neq 2$.

(b) We show that the partition dimension of $C_n *_2 K_m$ is $pd(C_n *_2 K_m) = 3$. For each $v \in V(C_n *_2 K_m)$, let $\Pi = \{S_1, S_2, S_3\}$ be the resolving partition where $S_1 = \{x, v_3\}$, $S_2 = \{y, v_4\}$, and $S_3 = \{u_3, \ldots, u_n\}$. We obtain the representation between all vertices of $V(C_n *_2 K_m)$ with respect to Π as follows

$r(x|\Pi) = (0, 1, 1)$, $r(y|\Pi) = (1, 0, 1)$, $r(u_j|\Pi) = (d(u_j, x), d(u_j, y), 0)$,
$r(v_3|\Pi) = (0, 1, 2)$, $r(v_4|\Pi) = (1, 0, 2)$,

for $3 \leq j \leq n$ with $d(u_j, x), d(u_j, y) \leq \lfloor \frac{n}{2} \rfloor$. Clearly that $r(x|\Pi) \neq r(y|\Pi) \neq r(u_3|\Pi) \neq \ldots \neq r(u_n|\Pi) \neq r(v_3|\Pi) \neq \ldots \neq r(v_m|\Pi)$, and then the partition dimension of $C_n *_2 K_m$ is $pd(C_n *_2 K_m) = 3$.

(2) Case $m \geq 5$.

Let $\Pi = \{S_1, S_2, \ldots, S_{m-1}\}$ be a resolving partition of $C_n *_2 K_m$ where $S_1 = \{x, v_3\}$, $S_2 = \{y, v_4\}$, $S_3 = \{u_3, \ldots, u_n\}$, and $S_i = \{v_{i+1}\}$, with $3 < i \leq m-1$.

The representations of each vertex of $C_n *_2 K_m$ with respect to Π are

$r(x|\Pi) = (0, 1, 1, \ldots, 1)$, $r(y|\Pi) = (1, 0, 1, \ldots, 1)$,
$r(u_3|\Pi) = (2, 1, 0, \ldots, d(u_3, v_i))$, $\ldots$,
$r(u_n|\Pi) = (1, 2, 0, \ldots, d(u_n, v_i))$, $r(v_3|\Pi) = (0, 1, 2, \ldots, 1)$,
$\ldots$, $r(v_m|\Pi) = (1, 0, 2, \ldots, 1)$,

with $d(u_3|\Pi), \ldots, d(u_n|\Pi) \leq \lfloor \frac{n}{2} \rfloor$. Therefore, for each vertex $v \in V(C_n *_2 K_m)$ has a distinct representation with respect to Π. So, the $C_n *_2 K_m$ graph has a resolving partition with $m-1$ elements. Next, we show that the cardinality of Π is $m-1$. The cardinality of Π is the number of $k-$partition in S_1, S_2, S_3 and S_i for $3 < i \leq m-1$. The number of $k-$partition in S_1, S_2 and S_3 are 1. The number of $k-$partition in S_i is $m-4$, so $|\Pi| = 1+1+1+(m-4) = m-1$. Therefore, $\Pi = \{S_1, S_2, \ldots, S_{m-1}\}$ is a resolving partition of $C_n *_2 K_m$.

We show that $C_n *_2 K_m$ has no resolving partition with $m-2$ elements. Assume Π is a resolving partition of $C_n *_2 K_m$ with $pd(C_n *_2 K_m) < m-1$ then for each vertex $v \in V(C_n *_2 K_m)$ has a distinct representation. If we choose $\Pi = \{S_1, S_2, \ldots, S_{m-2}\}$ as a resolving partition then one of the partition class contains 2 vertices v_i. By using Lemma 5.1, the distance of vertex x and y to vertex v_i, must be the same distance, and then vertex x and y belong to distinct of partition class. As a result, we have the same representation $r(x|\Pi) = r(y|\Pi)$, a contradiction. So, the partition dimension of $C_n *_2 K_m$ is $pd(C_n *_2 K_m) = m-1$ for $m \geq 5$. $\square$

6. Conclusion

According to the discussion above it can be concluded that the partition dimension of a lollipop graph $L_{m,n}$, a generalized Jahangir graph $J_{m,n}$, and a $C_n *_2 K_m$ graph are as stated in Theorem 3.1, Theorem 4.1, and Theorem 5.1, respectively.

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