

H-SUPERMAGIC COVERINGS ON SHRUBS GRAPH AND ON THE CORONA OF TWO GRAPHS

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Abstract. A finite simple graph G admits an H -covering if every edge of $E(G)$ belongs to a subgraph of G isomorphic to H . We said the graph $G = (V, E)$ that admits H -covering to be H -magic if there exists bijection function $f : V(G) \cup E(G) \rightarrow 1, 2, \dots, |V(G) + E(G)|$, such that for each subgraph H' of G isomorphic to H , $f(H') = \sum_{v \in V'} f(v) + \sum_{e \in E'} f(e) = m(f)$ is constant. Furthermore, if $f(V) = 1, 2, \dots, |V(G)|$ then G is called H -supermagic. In this research we defined $S_{2,2}$ -supermagic covering on shrub graph $\check{S}(m_1, m_2, \dots, m_n)$, moth-supermagic covering on $W_n \odot \bar{K}_m$ for even $n \geq 4$ and for prime $n \geq 5$, and fish-supermagic covering on $L_m \odot P_n$ for $m, n \geq 2$.

Keywords : $S_{2,2}$ -supermagic covering, moth-supermagic covering, fish-supermagic covering, shrub graph $\check{S}(m_1, m_2, \dots, m_n)$, $W_n \odot \bar{K}_m$ graph, $L_m \odot P_n$ graph

1. INTRODUCTION

Gallian [2] defined a graph labeling as an assignment of integers to the vertices or edges, or both, subject to certain condition. Graph labeling was first introduced in the late 1960s. Magic labeling is a type of graph labeling that the most often to be studied.

In 2005 Gutiérrez and Lladó [5] generalized the concept of an edge-magic total labeling into an H -magic covering as follows. Let $G = (V, E)$ be a finite simple graph that admits H -covering. A bijection function $f : V(G) \cup E(G) \rightarrow 1, 2, \dots, |V(G) + E(G)|$ is called H -magic labeling of G , for every subgraph $H' = (V', E')$ of G isomorphic to H , $f(H') = \sum_{v \in V'} f(v) + \sum_{e \in E'} f(e) = m(f)$ is constant. Graph G is called H -supermagic if $f(V(G)) = 1, 2, \dots, |V(G)|$. In [5], it is proved that a complete bipartite graph $K_{n,n}$ is $K_{1,n}$ -magic for $n \geq 1$.

Selvagopal and Jeyanthi in Gallian [2] proved that for a positive integer n , the k -polygonal snake of length n is C_4 -supermagic; for $m \geq 2, n = 3$, or $n > 4$, $C_n \times P_m$ is C_4 -supermagic; $P_2 \times P_n$ and $P_3 \times P_n$ are C_4 -supermagic for all $n \geq 2$. Roswitha et al. [9] proved H -magic covering on some classes of graphs. In this research we investigate that a shrub graph $\check{S}(m_1, m_2, \dots, m_n)$ admits a double star $S_{2,2}$ -supermagic labeling, $W_n \odot \bar{K}_m$ admits a moth-supermagic labeling, and $L_m \odot P_n$ admits a fish-supermagic labeling.

2. MAIN RESULTS

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2.1. k -balanced multiset. Maryati et al. [6] defined a multiset as a set that allows the same elements in it. Moreover, Maryati et al. [6] introduced a technique of partitioning a multiset, called k -balanced multiset, as follows.

Let $k \in \mathbb{N}$ and Y be a multiset that contains positive integers. Y is said to be k -balanced if there exists k subsets of Y , say Y_1, Y_2, \dots, Y_k , such that for every $i \in [1, k]$, $|Y_i| = \frac{|Y|}{k}$, $\sigma Y_i = \frac{\sigma Y}{k} \in \mathbb{N}$, and $\biguplus_{i=1}^k Y_i = Y$, then Y_i is called a balanced subset of Y .

Lemma 2.1. (Roswitha and Baskoro [8]) *Let x and y be non-negative integers. Let $X = [x+1, x(y+1)]$ with $|X| = xy$ and $Y = [x(y+2), 2x(y+1)-1]$ where $|Y| = xy$. Then, the multiset $K = X \uplus Y$ is xy -balanced with all its subsets are 2-sets.*

Proof. For every $j \in [1, xy]$, define $K_j = \{a_j, b_j\}$, where

$$\begin{aligned} a_j &= x+j, \\ b_j &= 2x(y+1)-j. \end{aligned}$$

Then, $\sum K_j = (2y+3)$ for each $j \in [1, xy]$, and so K is xy -balanced with all its subsets are 2-sets. \square

2.2. (k, δ) -anti balanced multiset. Inayah [3] defined (k, δ) -anti balanced multiset as follows. Let $k, \delta \in \mathbb{N}$ and X be a set containing the elements of positive integers. A multiset X is said to be (k, δ) -anti balanced if there exists k subsets from X , say X_1, X_2, \dots, X_k such that for every $i \in [1, k]$, $|X_i| = \frac{|X|}{k}$, $\biguplus_{i=1}^k X_i = X$, and for $i \in [1, k-1]$, $\sum X_{i+1} - \sum X_i = \delta$. Here, we give several lemmas on (k, δ) -anti balanced multiset.

Lemma 2.2. Let x, y, z and k be non-negative integers, $k \geq 2$. Let $R = [x, x + \lfloor \frac{k}{2} \rfloor] \uplus [x+1, x + \lfloor \frac{k}{2} \rfloor] \uplus [y, y + \lfloor \frac{k-1}{2} \rfloor] \uplus [y, y + \lfloor \frac{k-1}{2} \rfloor - 1] \uplus [z, z + k - 1]$, then R is $(k, 2)$ -anti balanced.

Proof. For every $j \in [1, k]$ we define the multisets $R_j = \{x + \lfloor \frac{j}{2} \rfloor, y + \lfloor \frac{j-1}{2} \rfloor, z + (j-1)\}$. It is obvious that for each $j \in [1, k]$, $|R_j| = 3$, $R_j \subset R$, and $\biguplus_{j=1}^k R_j = R$. Since $\sum R_j = x + y + z + \lfloor \frac{2j-1}{2} \rfloor + j - 1$ for every $j \in [1, k]$, then $\sum R_{j+1} - \sum R_j = 2$, R is $(k, 2)$ -anti balanced. \square

Lemma 2.3. Let x and k be non-negative integers, $k \geq 2$. Let $Y = [x, x+k] \uplus [x+1, x+k-1]$, then Y is $(k, 2)$ -anti balanced.

Proof. For every $j \in [1, k]$ we define the multisets $Y_j = \{x+j-1, x+j\}$. Now we have for every $j \in [1, k]$, $|Y_j| = 2$, $Y_j \subset Y$ and $\biguplus_{j=1}^k Y_j = Y$. Since $\sum Y_j = 2(x+k-j)+1$ for every $j \in [1, k]$, then $\sum Y_{j+1} - \sum Y_j = 2$, Y is $(k, 2)$ -anti balanced. \square

Here we provide two examples of those lemmas as follows.

- (1) Let $R = [2, 4] \uplus [3, 4] \uplus [3, 5] \uplus [3, 4] \uplus [4, 8]$, $x = 2$, $y = 3$, $z = 4$, and $k = 5$. By applying Lemma 2.2, we have 5-subsets of R as follows. $R_1 = \{2, 3, 4\}$, $R_2 = \{3, 3, 5\}$, $R_3 = \{3, 4, 6\}$, $R_4 = \{4, 4, 7\}$, and $R_5 = \{4, 5, 8\}$. It can be checked

that $\Sigma R_1 = 9, \Sigma R_2 = 11, \Sigma R_3 = 13, \Sigma R_4 = 15$, and $\Sigma R_5 = 17$ for every $i \in [1, 5]$. Therefore R is $(5, 2)$ -anti balanced.

- (2) Let $Y = [3, 6] \uplus [4, 5]$, $x = 3$ and $k = 3$. According to Lemma 2.3, we have 3-subsets of Y as follows. $Y_1 = \{3, 4\}$, $Y_2 = \{4, 5\}$, and $Y_3 = \{5, 6\}$. Hence $\Sigma Y_1 = 7, \Sigma Y_2 = 9$, and $\Sigma Y_3 = 11$ for every $i \in [1, 3]$. Therefore Y is $(3, 2)$ -anti balanced.

2.3. $S_{2,2}$ -supermagic Labeling on Shrubs Graphs. Maryati [6] defined a shrub graph $\check{S}(m_1, m_2, \dots, m_n)$ as a graph that obtained from a star graph $K_{1,n}$ for $n \geq 2$ with central vertex c , and adding some vertices and edges so that for every $i \in [1, n]$, v_i is related with the new $m_i \geq 1$ vertices. The double star $S(n, m)$ is the graph consisting of the union of two stars $K_{1,n}$ and $K_{1,m}$ together with a line joining their centers (Grossman [4]).

Theorem 2.1. *Any shrub graph $\check{S}(m_1, m_2, \dots, m_n)$ is $S_{2,2}$ -supermagic for any integer n and $m_i \geq 2, i \in [1, n]$.*

Proof. Let G be a shrub graph $\check{S}(m_1, m_2, \dots, m_n)$ for any integer n and $m_1, m_2, \dots, m_n \geq 2$. Then $|V(G)| = m_1 + m_2 + \dots + m_n + n + 1$ and $|E(G)| = m_1 + m_2 + \dots + m_n + n$. Let $A = [1, 2(n + m_1 + m_2 + \dots + m_n) + 1]$. Partition A into 3 sets, $A = K \cup X$, with $X = L \uplus M$ where $K = \{1\}$, $L = [2, n + 1 + m_1 + \dots + m_n]$, $M = [n + 2 + m_1 + \dots + m_n, 2n + 1 + 2(m_1 + \dots + m_n)]$. Let f be a total labeling of G and $s(f)$ be the supermagic sum in every subgraph H of G that is isomorphic to double star $S_{2,2}$. Now we define a total labeling f on G as follows. Label the center vertex with 1. Apply Lemma 2.1 to partition X into $\{X_i, x \leq i \leq y\}$ with $x = 1$ and $y = n + m_1 + m_2 + \dots + m_n$. We obtain that X is $n + m_1 + m_2 + \dots + m_n$ -balanced with all its subsets are 2-sets and $\sum X_i = 2(n + m_1 + m_2 + \dots + m_n) + 3$. Use the smaller labels in every X_i for the vertices. Thus, the sum of labels of each double star $S_{2,2}$ is $s(f) = 10(n + m_1 + m_2 + \dots + m_n) + 16$. Hence, the shrubs graph is $S_{2,2}$ -supermagic. \square

2.4. Moth-Supermagic Labeling on $W_n \odot \overline{K}_m$. $W_n \odot \overline{K}_m$ graph is obtained by taking one copy of W_n and $n + 1$ copies of \overline{K}_m and then joining by a line the i 'th vertex of W_n to every vertex in the i 'th copy of \overline{K}_m . The definition of a moth graph is $K_1 + \{P_3 \cup 2K_1\}$ (Menon [7]).

Theorem 2.2. *For even $n \geq 4$ and for prime $n \geq 5$, $W_n \odot \overline{K}_m$ graph is a moth-supermagic.*

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Proof. Let G be $W_n \odot \overline{K}_m$. Let $V(C_n) = \{v_1, v_2, v_3, \dots, v_n\}$, $V(\overline{K}_m) = \{u_1, u_2, u_3, \dots, u_{m(n+1)}\}$, and v_0 be the central vertex of G . We obtain that $V(G) = V(C_n) \cup V(\overline{K}_m) \cup \{v_0\}$ and $E(G) = \{v_0 v_1, v_0 v_2, \dots, v_0 v_n, v_1 v_2, v_2 v_3, \dots, v_{n-1} v_n, v_0 u_1, v_0 u_2, \dots, v_0 u_m, \dots\}$.

$v_0v_{m+1}, \dots, v_0v_{2m}, \dots, v_nv_{nm+1}, \dots, v_{(n+1)m}$. Let $L_i = \{v_i, v_iv_0\}$ for $i \in [1, n]$, $M_j = \{u_j, u_jv_0\}$ for $j \in [1, m]$ and $N_{ik} = \{u_k, u_kv_i\}$ for $k \in [m+1, (n+1)m]$ and $i \in [1, n]$. Let f be a total labeling of G and $s(f)$ be the supermagic sum in every subgraph H of G that is isomorphic to moth graph. The supermagic sum is obtained by the formula

$$s(f) = f(L_i) + f(L_{i+1}) + f(L_{i+2}) + f(v_iv_{i+1}) + f(v_{i+1}v_{i+2}) + 2f(N_{ik}/M_j) + f(v_0) \quad (2.1)$$

Furthermore, we divide the proof into two cases according to the values of n .

Case 1. n is even ($n \geq 4$).

The total labeling f of G is defined as follows.

- (1) Label each vertex of G other than central vertex by

$$f(v_i) = \begin{cases} \frac{i+1}{2} + 1, & \text{for } i \text{ odd; } i \in [1, n] \\ \frac{2n+4-i}{2}, & \text{for } i \text{ even; } i \in [1, n]. \end{cases}$$

- (2) Label each edge of G as follows

$$f(v_0v_i) = \begin{cases} 5(n+1) + \frac{n}{2}, & \text{for } i = 2 \\ 5(n+1) + \frac{i-2}{2}, & \text{for } i \text{ even; } i \in [4, n] \\ 6(n+1) - \frac{i+1}{2}, & \text{for } i \text{ odd; } i \in [1, n] \end{cases}$$

and

$$f(v_iv_{i+1}) = \begin{cases} 6(n+1) + \frac{n}{2}, & \text{for } i = 2 \\ 7(n+1) - \frac{i}{2}, & \text{for } i \text{ even; } i \in [4, n] \\ 6(n+1) + \frac{i-1}{2}, & \text{for } i \text{ odd; } i \in [1, n]. \end{cases}$$

Thus, $f(v_0) = 1, f(L_n) = 6(n+1), f(L_1) = 6(n+1) + 1, f(L_2) = 6(n+1) + \frac{n}{2}, f(v_nv_1) = 7(n+1) - \frac{n}{2}$, and $f(v_1v_2) = 6(n+1)$. From Lemma 2.1 with $X = [x+2, 3(x+1)]$ and $Y = [3y+1, 5y], x = n, y = n+1$ we obtain that $f(N_{ik}) = 6(n+1) + 1$. Let $s(f)$ be a constant sum in every subgraph moth graph of G . By Equation (2.1) we have

$$s(f) = 43(n+1) + 4$$

Hence, $W_n \odot \overline{K}_m$ for n even is a moth-supermagic.

Case 2. n is prime ($n \geq 5$).

The total labeling f of G is defined as follows.

- (1) Label each vertex of G other than central vertex by

$$f(v_i) = \begin{cases} 2, & \text{untuk } i = 1 \\ 2 + \lfloor \frac{i}{3} \rfloor, & \text{for } i \bmod 3 = 1, i \in [4, n]. \end{cases}$$

For $n \bmod 3 = 2$

$$f(v_i) = \begin{cases} \frac{n+i+5}{3}, & \text{for } i \in \{2, 5, 8, \dots, n\} \\ \lfloor \frac{2n+i}{3} \rfloor + 2, & \text{for } i \in \{3, 6, 9, \dots, n-2\}. \end{cases}$$

For $n \bmod 3 = 1$

$$f(v_i) = \begin{cases} \frac{2n+i+5}{3}, & \text{for } i \in \{2, 5, 8, \dots, n-2\} \\ \frac{n+i+5}{3}, & \text{for } i \in \{3, 6, 9, \dots, n-1\}. \end{cases}$$

(2) Label each edge of G as follows

$$f(v_0 v_i) = \begin{cases} 5(n+1) + 1, & \text{for } i = n-2 \\ f(v_0 v_{(2i+n+3) \bmod n}) + 1, & \text{for } i \neq n-2, i \neq \frac{n-3}{2}; i \in [1, n] \\ f(v_0 v_{(\frac{2i+n+3}{2})}) + 1, & \text{for } i = \frac{n-3}{2}. \end{cases}$$

$$f(v_i v_{i+1}) = 6(n+1) + \frac{i - \frac{n-3}{2}}{2}, \quad \text{for } i \in \{\frac{n-3}{2}, \frac{n-3}{2} + 2, \frac{n-3}{2} + 2 + 2, \dots, n-1\}$$

For $(\frac{n-3}{2})$ is even.

$$f(v_i v_{i+1}) = \begin{cases} 6(n+1) + \lceil \frac{n+2i+2}{4} \rceil, & \text{for } i \text{ odd}; i \in [1, n] \\ 6(n+1) + \lceil \frac{3n+2}{4} \rceil + \frac{i}{2}, & \text{for } i \text{ even}, n > 7; i \in [2, \frac{n-7}{2}]. \end{cases}$$

For $(\frac{n-3}{2})$ is odd.

$$f(v_i v_{i+1}) = \begin{cases} 6(n+1) + \lceil \frac{3n+2i}{4} \rceil, & \text{for } i \text{ odd}, n > 5; i \in [1, \frac{n-3}{2}], \\ 6(n+1) + \lceil \frac{n+2i}{4} \rceil, & \text{for } i \text{ even}; i \in [2, n]. \end{cases}$$

For $f(v_0) = 1$. By Lemma 2.1 with $X = [x+2, 3(x+1)]$ and $Y = [3y+1, 5y]$, $x = n$, $y = n+1$ we obtain that $f(N_{ik}) = 6(n+1) + 1$. Let $s(f)$ be a constant sum in every subgraph moth graph of G . By Equation (2.1) we have

$$s(f) = 42(n+1) + (n-3) + 8$$

Hence, $W_n \odot \overline{K_m}$ when n prime is a moth-supermagic. This completes the proof of the theorem. \square

2.5. Fish-Supermagic Labeling on $L_m \odot P_n$. $L_m \odot P_n$ graph is obtained by taking one copy of L_m and $2(m+1)$ copies of P_n and then joining by a line the i 'th vertex of L_m to every vertex in the i 'th copy of P_n .

Theorem 2.3. Any $L_m \odot P_n$ graph for $m, n \geq 2$ is fish-supermagic.

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Proof. Let G be the $L_m \odot P_n$ graph for any integer $m, n \geq 2$. Then $|V(G)| = 2(m+1)(n+1)$ and $|E(G)| = 2(m+1)(2n-1) + 3m+1$. Let $A = [1, 2(m+1)(3n-1) + 5m+3]$. We define a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, 2(m+1)(3n-1) + 5m+3\}$.

Let $A = [1, 2(m+1)(3n-1) + 5m + 3]$. Partition A into 4 sets, $A = V \uplus W \uplus X \uplus Y$, where $V = [1, 2n(m+1)]$, $W = [(m+1)(2n+3), (m+1)(4n+3) - 1]$, $X = [(m+1)(4n+3), (m+1)(4n+3) + n - 2] \uplus [(m+1)(4n+3) + n, (m+1)(4n+3) + (2n-2)] \uplus \dots \uplus [(m+1)(4n+3) + (2m+1)n, (m+1)(4n+3) + 2n(m+1) - 2]$, and $Y = [2n(m+1) + 1, 2(n+1)(m+1) + m] \uplus \{(m+1)(4n+3) + (n-1), (m+1)(4n+3) + (n-1) + n, (m+1)(4n+3) + (n-1) + 2n, \dots, (m+1)(4n+3) + (n-1) + 2mn\}$.

Let f be a total labeling of G and $s(f)$ be the supermagic sum in every subgraph H of G that is isomorphic to fish graph. Brandstadt [1] defined a fish graph as a graph on 6 vertices illustrated bellow.

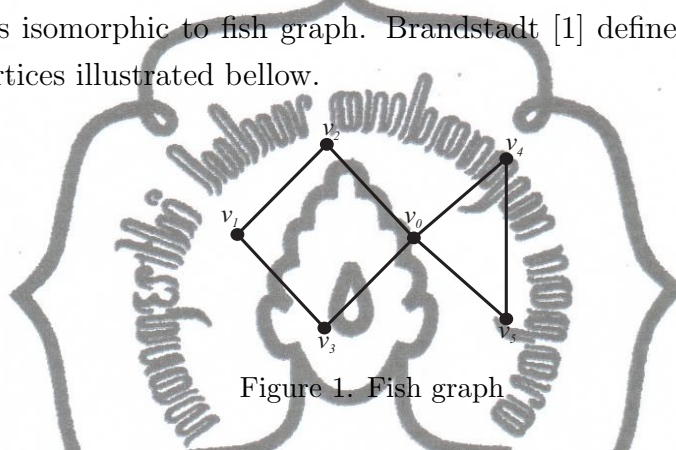


Figure 1. Fish graph

It is easier if fish graph was divided into 2 subgraphs, $F_2 - \{v_0\}$ (Fan graph F_2 without central vertex) and C_4 . There are two steps to label the $L_m \odot P_n$ graph. The first step is to prove that $(2m+2)(F_n - \{v_0\})$ is $(F_2 - \{v_0\})$ -supermagic using the element of multisets $V \uplus W \uplus X$, and the second one is to prove that L_m is C_4 -supermagic using the element of set Y .

Step 1. The first step is to prove that $(2m+2)(F_n - \{v_0\})$ is $(F_2 - \{v_0\})$ -supermagic using the element of multisets $V \uplus W \uplus X$. Furthermore we divide into two cases.

Case 1. $n = 2$.

For $n = 2$, the number of $(F_2 - \{v_0\})$ subgraphs on graph G is $2m+2$ as shown in Figure 2.

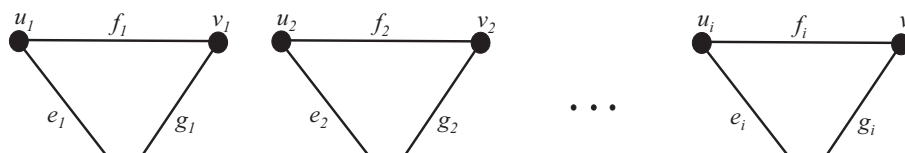


Figure 2. $F_2 - \{v_0\}$ subgraph on $L_m \odot P_2$

Now, we prove that $(2m+2)(F_2 - \{v_0\})$ on $L_m \odot P_2$ is $(F_2 - \{v_0\})$ -supermagic using the element of multisets $V \uplus W \uplus X$. Let H be $(2m+2)(F_2 - \{v_0\})$. We have $V(H) = \{u_i, v_i : 1 \leq i \leq 2(m+1)\}$ and $E(H) = \{e_i \cup f_i \cup g_i : 1 \leq i \leq 2(m+1)\}$. Let f be the total labeling of H and $s(f)$ be the supermagic sum in every $F_2 - \{v_0\}$ subgraph on $(2m+2)(F_2 - \{v_0\})$. The total labeling of H for every $1 \leq i \leq 2(m+1)$ is defined as follows.

$$f(x) = \begin{cases} i, & \text{if } x = v_i, \\ 2(m+1) + i, & \text{if } x = u_i, \\ (m+1)(4n+3) + 1 - 2i, & \text{if } x = e_i, \\ (m+1)(4n+3) + 2(i-1), & \text{if } x = f_i, \\ (m+1)(4n+3) - 2i, & \text{if } x = g_i. \end{cases}$$

For every $1 \leq i \leq 2(m+1)$, let $(F_2 - \{v_0\})^i$ be the subgraph of $L_m \odot P_2$ with $V((F_2 - \{v_0\})^i) = \{u_i, v_i\}$ and $E((F_2 - \{v_0\})^i) = \{e_i, f_i, g_i\}$. It can be checked that for every $1 \leq i \leq 2(m+1)$, $\sum f((F_2 - \{v_0\})^i) = 3(m+1)(4n+3) + 2(m+1) - 1$. Hence $(2m+2)(F_2 - \{v_0\})$ is $(F_2 - \{v_0\})$ -supermagic.

Case 2. $n > 2$.

Now we prove that $(2m+2)(F_n - \{v_0\})$ is $(F_2 - \{v_0\})$ -supermagic using the element of multisets $V \uplus W \uplus X$. Let f be the total labeling of $(2m+2)(F_n - \{v_0\})$ and $s(f)$ be the supermagic sum in every $F_2 - \{v_0\}$ of $(2m+2)(F_n - \{v_0\})$. The total labeling of $(F_n - \{v_0\})^i$ for every $1 \leq i \leq 2(m+1)$ will be executed within two steps. Firstly, we label the subgraph of $(F_n - \{v_0\})^i$ called P_n^i , then we label the edges besides on P_n^i that are incident with the vertices of P_n^i .

Lemma 2.2 is applied for every i of P_n^i where $1 \leq i \leq 2(m+1)$ with $x_i = 1 + (i-1)\lceil \frac{n}{2} \rceil$, $y_i = 2(m+1)\lceil \frac{n}{2} \rceil + (i-1)\lfloor \frac{n}{2} \rfloor + 1$, $z_i = (m+1)(4n+3) + n(i-1)$, and $k = n-1$ using the element of $V \uplus X$. We have $V \uplus X$ in every P_i is $(k, 2)$ -anti balanced. Next, by applying Lemma 2.3 we label the edges besides on P_n^i that are incident with the vertices of P_n^i using the element of W with $x = (m+1)(4n+3) - ni$ and $k = n$, starting from the edge that incident with a vertex v_n until the edge that incident with a vertex v_1 in every P_n^i . Now we have that W for every $i \in [1, 2(m+1)]$ is $(k, 2)$ -anti balanced. This implies that multiset $V \uplus W \uplus X$ is $2k(m+1)$ -balanced. Hence $(2m+2)(F_n - \{v_0\})$ of $L_m \odot P_n$ is $(F_2 - \{v_0\})$ -supermagic with $\sum f((F_2 - \{v_0\})^i) = (m+1)(12n + 2\lceil \frac{n}{2} \rceil + 2) + 7m + 6$. We conclude that for $n \geq 2$, $(2m+2)(F_n - \{v_0\})$ is $(F_2 - \{v_0\})$ -supermagic.

Step 2. The next step is to prove that L_m is C_4 -supermagic using the element of set Y . Let $L_m \cong P_m \times P_2$ be a graph with $V(L_m) = \{u_i, v_i : 1 \leq i \leq m+1\}$ and $E(L_m) = \{u_i v_i : 1 \leq i \leq m+1\} \cup \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq m\}$. Let f be a total labeling of $V(L_m) \cup E(L_m)$ using the element of set Y . The total labeling of L_m is defined as follows.

Label each vertex of L_m as follows.

$$f(u_i) = (m+1)(2n+1) - (i-1)m, \text{ for } i \in [1, m+1].$$

$$f(v_i) = \begin{cases} (m+1)(2n+1) + \frac{i+1}{2}, & \text{for } i \text{ odd, } i \in [1, m+1], \\ (m+1)(2n+1) + \lceil \frac{m+1}{2} \rceil + \frac{i}{2}, & \text{for } i \text{ even, } i \in [1, m+1]. \end{cases}$$

Next, label each edge of L_m for every $i \in [1, m]$ as follows.

$$\begin{aligned} f(u_i u_{i+1}) &= (m+1)(4n+3) - 1 + ni \\ f(v_i v_{i+1}) &= (m+1)(2n+2) + i \end{aligned}$$

For every $i \in [1, m+1]$, label each edge of L_m as follows.

For m odd.

$$f(u_i v_i) = \begin{cases} (m+1)(4n+3) + n(m+1) + mn - 1 - n(\frac{i-1}{2}), & \text{for } i \text{ odd,} \\ (m+1)(4n+3) + n(\frac{m+1}{2}) + mn - 1 - n(\frac{i-2}{2}), & \text{for } i \text{ even.} \end{cases}$$

For m even.

$$f(u_i v_i) = \begin{cases} (m+1)(4n+3) + n(\frac{m+2}{2}) + mn - 1 - n(\frac{i-1}{2}), & \text{for } i \text{ odd,} \\ (m+1)(4n+3) + n(m+1) + mn - 1 - n(\frac{i-2}{2}), & \text{for } i \text{ even} \end{cases}$$

For every $1 \leq i \leq m$, C_4^i be the subgraph of $L_m \odot P_n$ with $V(C_4^i) = \{u_i, u_{i+1}, v_i, v_{i+1}\}$ and $E(C_4^i) = \{u_i u_{i+1}, v_i v_{i+1}, u_i v_i, u_{i+1} v_{i+1}\}$. It can be checked that for every $1 \leq i \leq m$, $\sum f(C_4^i) = (m+1)(22n+14) + n(2 + \lceil \frac{m+1}{2} \rceil) + \lceil \frac{m+1}{2} \rceil + m(3n+1)$. Hence L_m is C_4 -supermagic.

It has been proved that $(2m+2)(F_n - \{v_0\})$ is $(F_2 - \{v_0\})$ -supermagic and L_m is C_4 -supermagic. Hence, $L_m \odot P_n$ is fish-supermagic with its supermagic sum is $s(f) = (m+1)[(12n+2+2\lceil \frac{n}{2} \rceil) + (22n+14)] + n(2 + \lceil \frac{m+1}{2} \rceil) + \lceil \frac{m+1}{2} \rceil + m(3n+8) + 6$. \square

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