AN INTEGRATED INVENTORY MODEL WITH TRANSPORTATION IN A DIVERGENT SUPPLY CHAIN AND UNSTABLE LEAD TIME AND SETUP COST

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ABSTRACT. This research presents single-vendor multi-buyers model with unstable lead time and setup cost and transportation in divergent supply chain. The vendor’s concern is on crashing of the setup cost, and buyers is on reducing lead time. The vendor manufactures products and delivers them to the buyers located in different locations by a fleet of vehicles with identical capacity. The lead time demand is normally distributed. Excluding transportation time, the lead time component of buyers can be reduced by adding crashing cost. Lead time demand per unit time on buyers are normally distributed. The purpose of this research is to formulate an integrated single-vendor multi-buyers model to determined the optimal solution of order quantity, safety factor, lead time, shipment frequency and routing decision which has been illustrated through a numerical example.

Keyword: integrated inventory model, unstable lead time, unstable setup cost, service level constraint, partial backorder

1. INTRODUCTION

Inventory management is main component in order a company can run well. Inventory management is used to find the right amount of product’s quantity and the right timing to order products in order to reduce holding cost. At first inventory management is managed separately by vendor and buyer, but in recent years many researchers developed integrated inventory management model by vendor and buyers. Goyal [1] is the first researcher who developed integrated vendor-buyer inventory model.

In integrated model there will be lead time from buyers order until products arrived. Ben-Daya and Raouf [2] added crashing cost into model in order to reduce lead time. Winston [3] explained, it is possible for buyers to order in lead time period. If the extended order is bigger than vendor’s supplies then there will be stockout. There are three kinds of order which happened during stockout. First,
called backorder when all buyers willing to wait product’s arrival. Second, called lost sales when all buyers refused to wait product’s arrival, and third called partial backorder when some buyers willing to wait and the rest refused. Ouyang and Wu [4] added shortage cost into model. Shortage cost is cost which emerged due to loss of profits and decreasing costumers level credibility then resulting loss of costumers. Shortage cost is hard to estimate because it’s hard to estimate loss of profits which are effects from product stockout and decreasing costumer level credibility, so that researchers (Jha and Shanker [5], Ouyang and Wu [4], Wei and Qiu [6]) replace shortage cost with service level constraint.

Jha and Shanker [5] described integrated inventory single-vendor multi buyers model with transportation in divergent supply chain under service level constraint. It’s assumed that all buyers willing to wait. After products had been finished manufactured then distributed to buyers in several routes with identical capacity vehicles.

This research developed an integrated inventory with transportation in divergent supply chain under service level constraint refers to Jha and Shanker [5] as well as unstable lead time and setup cost refers to Jaggi and Arneja [7] on partial backorder case refers to Ni et al [8]. Furthermore, the optimal order quantity, safety factors, lead time of buyers and shipment frequency per production cycle, setup cost of vendor and efficient routes are determined simultaneously by solving the combined integrated inventory problem and vehicle routing problem (VRP) using a coordinated two-phase iterative approach.

2. NOTATION AND ASSUMPTIONS

The following notations and assumptions are used to defined the problem. Some additional notations and assumptions will be listed later when they are needed.
2.1 Notations

\( N \) number of buyers indexed from 1 to \( N \), index 0 denotes vendor

\( E(.) \) mathematical expectation

\( x^+ \) maximum value of \( x \) and 0, i.e. \( x^+ = \max\{x, 0\} \)

**For the \( i \) – th buyer (\( i = 1, 2, ..., N \))**

\( D_i \) average demand per unit time

\( A_i \) ordering cost per order

\( C_{bi} \) unit purchase cost

\( h_i \) holding cost rate per unit time

\( r_i \) reorder point

\( Q_i \) order quantity (decision variable)

\( k_i \) safety factor (decision variable)

\( L_i \) lead time (decision variable)

\( T_i \) transportation time for an order to arrive at buyer \( i \) from the vendor (decision variable)

\( \alpha_i \) proportion of demand that are not met from stock so \( (1 - \alpha_i) \) is the service level.

\( B_i(r_i) \) expected demand shortages at the end of buyer’s cycle

\( X_i \) lead time demand, which is normally distributed with finite mean \( D_i L_i \) and standard deviation \( \sigma_i \sqrt{L_i} \), where \( \sigma_i \) denotes the standard deviation of demand per unit time, \( X_i \sim N(D_i L_i, \sigma_i \sqrt{L_i}) \)

**For the vendor**

\( P \) production rate, \( P > D(D = \sum_{i=1}^{N} D_i) \)

\( A_v \) setup cost per setup (decision variable)

\( C_v \) unit production cost (\( C_v < C_{bi}, \forall i \))

\( h_v \) holding cost rate per unit time

\( m \) number of lots delivered from the vendor to each buyer (shipment frequency) in a production cycle, which is same for all buyers, a positive integer (decision variable)

\( Q \) shipment lot size per trip of all vehicles, \( Q = \sum_{i=1}^{N} Q_i \)
For vehicle routing

\( K \) number of routes indexed from 1 to \( K \) (decision variable)
\( R_g \) route \( g: 0 - i_g(1) - i_g(2) - \cdots - i_g(p_g) - 0 \), where \( i_g(f) \) is the index of the \( j \)-th buyer visited and \( p_g(1 \leq p_g \leq N) \) is the number of buyers in the \( g \)-th route. Every route starts and finishes at the vendor (decision variable)
\( l_g \) length of route \( g, g(=1, 2, \ldots, K) \)
\( f \) fixed cost of using a vehicle on a route
\( C \) variable cost per unit distances
\( b \) vehicle capacity
\( S \) average speed of vehicle
\( d_{ij} \) distances between location \( i \) and \( j, d_{ij}(\forall i, j = 0, 1, \ldots, N) \).

Figure 1. The inventory pattern for the vendor and buyer i.
2.2 Assumptions

1. Buyers $i$ orders a lot size $Q_i (1 = 1, 2, 3, ..., N)$ and the vendor manufactures $mQ$ units with a finite production rate $P (P > D)$ in one setup but ship in quantity $Q = \sum_{i=1}^{N} Q_i$ over $m$ times using identical capacity vehicles to meet the demands of all the buyers. The vehicles on all routes are dispatched simultaneously at the interval of the buyer’s common average ordering cycle time, i.e. $\frac{Q_1}{D_1} = \frac{Q_2}{D_2} = \cdots = \frac{Q_N}{D_N} \Rightarrow Q_i = \frac{QD_i}{D}$.

2. Each buyer reviews inventory using continuous reviews policy and places an order whenever inventory level falls to the reorder point $r$ (Figure 1.). The reorder point $r$ is equal to expected demand during lead time plus safety stock, that is $r_i = D_i L_i + k_i \sigma_i \sqrt{L_i}$ where $k_i$ is safety factor for buyer $i$.

3. The setup cost $A_v$ for the vendor consists of $u$ mutually independent components. The $j$-th component has a normal cost $e_j$ and minimum cost $d_j$ and a crashing cost $s_j$ when the normal cost reduces to minimum cost. Arranging $s_j$ such that $s_1 \leq s_2 \leq \cdots \leq s_u$, crashing of the setup cost starts from its first component as it acquires the minimum unit crashing cost, then the component 2 and so on.

4. Let $A_{v0} = \sum_{j=1}^{m} e_j$ be the total normal setup cost without crashing and $A_{vj}$ be the reduced setup cost when $j$ component crashed to their minimum cost, where $j = 1, 2, 3, ..., u$ and given $A_{vj} = A_{v0} - \sum_{t=1}^{j} (e_t - d_t)$, $t = 1, 2, ..., u$ and setup crashing cost per cycle $R(A_{vj})$ is given as $R(A_{vj}) = \sum_{t=1}^{j} s_t$.

5. The lead time $L_i$ of buyer $i$ has $n_i + 1$ mutual independent components. The first $n_i$ components are controllable while the $(n_i + 1)$th component is fixed transportation time from the vendor to buyer $i$. The $r$-th controllable component of lead time of buyer $i$ has a minimum duration.
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\(a_{i,r}\), normal duration \(b_{i,r}\) and a crash cost per unit time \(c_{i,r}\). Further, without loss of generality, we assume that \(c_{i,1} \leq c_{i,2} \leq \cdots \leq c_{i,n_i} \forall i.\)

6. The controllable lead time component of each buyer is crashed one at a time starting with the least crashing cost \(c_{i,r}\), \(\forall i\) component and so on.

7. Let \(L_{i0} = \sum_{j=1}^{n_i} b_{i,j} + \sum_{j=1}^{r_i} a_{i,j} + T_i, \ r = 1, 2, \ldots, n_i\) and \(L_{ir}\) be the length of lead time for buyer \(i\) with components \(1, 2, \ldots, r\) crashed to their minimum duration, the length of lead time crashing cost \(C_i(L_i)\) per cycle of the \(i\) th buyer for a given \(L_i \in [L_{ir}, L_{ir-1}]\) is given by \(C_i(L_i) = c_{ir}(L_{ir-1} - L_i) + \sum_{j=1}^{r_i-1} c_{i,j}(b_{i,j} - a_{i,j}), \ \forall i.\)

8. If a shortened lead time is requested by a buyer then the extra cost incurred by the vendor will be fully transferred to that buyer. Therefore, lead time crashing cost is a cost component of the buyer.

9. Loading/unloading time is included in the travel time between vendor – buyers.

3. MODEL FORMULATION

This section will explain about model formulation of total expected cost per unit time for buyers, total expected cost per unit time for vendor, total transportation cost per unit time and joint total expected cost per unit time.

3.1 Total expected cost for vendor. Vendor manufactured the item in quantity of \(mQ\) in one production cycle and delivered and each buyer will receive it in \(m\) lots each size \(Q_i\)‘s such that \(Q_i = D_iQ/D\) for \(i = 1, 2, \ldots, N\). Therefore, the expected length of production cycle for vendor is \(mQ/D\). The total expected cost of the vendor consists of production setup, crashing setup cost and inventory holding cost

\[TEC_v(Q, A_v, m) = \frac{A_vD}{m} + \frac{h_vC_vQ}{2} \left[t \left(1 - \frac{D}{P} \right) - 1 + \frac{2D}{P} \right] + \frac{R(A_v)}{mQ} \quad (3.1)\]

3.2 Total expected cost for buyer. If the expected cycle time of \(Q_i/D_i\) for the buyer \(i\), then the ordering cost per unit time is \(A_iD_i/Q_i\) and the lead time crashing cost per unit time is \(D_iC_i(L_i)/Q_i\). The expected net inventory level for buyer \(i\) just before receipt of an order is \(r_i - D_iL_i\) and the expected net inventory level after receipt of an order is \(Q_i + r_i - D_iL_i\). Hence, the average inventory over the cycle for buyer \(i\) can be approximated to \((Q_i/2) + k_i\sigma_i\sqrt{L_i}\), where safety factor \(k_i\) satisfies probability that the lead time demand at buyer \(i\) exceeds reorder point \(r_i\). If the lead time demand is assumed normal distribution with mean \(D_iL_i\) and standard deviation \(\sigma_iL_i\) then the expected shortage at the end of the cycle is
\[ E(X_i - r_i) = \sigma_i \sqrt{L_i \varphi(k_i)} \] where \( \varphi(k_i) = \phi(k_i) - k_i [1 - \psi(k_i)] \), \( \varphi \) and \( \psi \) respectively are standard normal pdf and CDF, so the expected number of backorders per cycle is \( \beta_i \sigma_i \sqrt{L_i \varphi(k_i)} \) and expected number of lost sales is \( (1 - \beta_i) \sigma_i \sqrt{L_i \varphi(k_i)} \). Therefore, the expected holding cost per year is

\[ h_i c_{bi} [D_i/2 + k_i \sigma_i \sqrt{L_i} + (1 - \beta_i) \sigma_i \sqrt{L_i \varphi(k_i)}]. \]

So, with total buyer \( i \), \( Q_i = D_i Q / D \), the total expected cost per unit time for the \( i \)th buyer consists of ordering cost, holding cost and lead time crashing cost.

\[ TEC_{bi}(Q_i, L_i) = \frac{A_i D}{Q} + h_i c_{bi} \left[ \frac{Q_i D_i}{2D} + (k_i \sigma_i \sqrt{L_i}) \right] + \frac{D_i}{Q} c_i(L_i) \] (3.2)

### 3.3 Total expected transportation cost per unit time.

The item produced as the vendor is delivered to the buyers using identical capacity vehicles. The vehicles combine the deliveries of several buyers into efficient routes and dispatched simultaneously on all the routes at a common average ordering intervals. The routing cost of a vehicle consists of a fixed cost which is incurred each time a tour initiated and a variable cost dependent on the distance traveled. The fixed cost may include the vehicle rental cost or any other fixed cost which do not depend on the load size, the route, the number of stops on the route, etc. (Jha and Shanker [5]).

The expected transportation cost per unit time is

\[ TC = \frac{D}{Q} (K f + C \sum_g g_i). \] (3.3)

### 3.4 Joint total expected cost per unit time.

The joint total expected cost per unit time for the vendor-buyers integrated system is the sum of total expected cost of the vendor, total expected cost of the buyer and expected transportation cost so the problem to be solved is to minimized

\[ JTEC(Q, k_1, k_2, ..., k_N, L_1, L_2, ..., L_N, A_v, m) = \frac{D}{Q} \left[ \frac{A_v + E(A_g)}{m} \right] + \sum_{i=1}^{N} \left( A_i + C_i(L_i) \right) + (K f + C \sum_{g=1}^{g_i} g_i) + \sum_{i=1}^{N} h_i c_{bi} [D_i Q / 2D + (1 - \beta_i) \sigma_i \sqrt{L_i \varphi(k_i)}] + \frac{Q}{2} h_v c_v \left[ m \left( 1 - \frac{D}{P} \right) - 1 + \frac{2D}{P} \right] \] (3.4)

### 3.5 Service Level Constraint

Shortage cost is cost which emerged due to loss of profits and decreasing costumers’ level credibility then resulting loss of costumers. Shortage cost is hard to estimate because it’s hard to estimate loss of profits which are effects from product stockout and decreasing costumer level credibility, so that researchers (Jha and Shanker [5], Ouyang and Wu [4], Wei and Qiu [6]) replace shortage cost with service level constraint. SLC puts a limit on the proportion of demands not met from stocks, which should not exceed a specified value. The SLC for buyer \( i \) can be obtained as the ratio of expected demand shortages at the end of cycle and quantity available for satisfying the demand per cycle. We get SLC for buyer \( i \) as
\[
\frac{d\sigma_j}{dQ} \leq \alpha_i
\]  
(3.5)

4. SOLUTION TECHNIQUE

4.1 Solving The Integrated Inventory Problem

To find minimum total expected cost of \( JT \) we have to find optimal solution from variable decision \( Q, k_i, L_i, A_v, m \). Temporarily ignored service level constrain (SLC) on all buyers from the equation (3.2) and assumed that vehicle routes are known, which makes the transportation cost component of the total expected cost as a function of \( Q \) similar to the ordering cost component of the buyers. For fixed \( Q, A_v, m \) and \( L_i \in [L_{i,r}, L_{i,r-1}], \forall i \), the joint total expected cost per unit time will have minimum value when safety factor of all buyers are zero. Since the function \( JT \) is linear with respect to \( A_v \), it could be taken as concave as well as convex too. Therefore, \( A_v \) can be treated as fixed cost item. Now, for fixed \((k_i, L_i, A_v, m)\) take partial derivatives of \( JT \) with respect to \( m \) and \( L_i \in [L_{i,r}, L_{i,r-1}], \forall i \) respectively, and obtain

\[
\frac{\partial JT}{\partial Q} (Q,k_1,k_2,\ldots,k_n,L_1,L_2,\ldots,L_n,A_v,m) = -\frac{D}{Q} \left( \frac{A_v + R(A_v)}{m} \right) + \frac{1}{2D} \sum_{i=1}^{N_i} h_i C_{bi} D_i + \frac{h_v C_v}{2} \left[ m \left(1 - \frac{D}{p}\right) - 1 + \frac{2D}{p} \right] 
\]  
(4.1)

and

\[
\frac{\partial JT}{\partial L_i} (Q,k_1,k_2,\ldots,k_n,L_1,L_2,\ldots,L_n,A_v,m) = -\frac{D}{Q} c_{i,r} \sum_{i=1}^{N_i} h_i C_{bi} k_i \sigma_i L_i^{-1/2} 
\]  
(4.2)

Hence, for fixed \((L_i, m)\), \( JT \) is convex in \( Q \) since

\[
\frac{\partial^2 JT}{\partial Q^2} (Q,k_1,k_2,\ldots,k_n,L_1,L_2,\ldots,L_n,A_v,m) = \frac{2D}{Q^3} \left( \frac{A_v + R(A_v)}{m} \right) + \sum_{i=1}^{N_i} \left( A_i + C_i(L_i) \right) + \left( Kf + C \sum_{g=1}^{K} l_g \right) > 0
\]

However, for fixed \((Q, m)\), \( JT \) is concave in \( L_i \) because

\[
\frac{\partial^2 JT}{\partial L_i^2} (Q,k_1,k_2,\ldots,k_n,L_1,L_2,\ldots,L_n,A_v,m) = -\frac{h_i}{4} C_{bi} k_i \sigma_i L_i^{-3/2} < 0, \forall i
\]

Therefore, for fixed \((Q, m)\), the minimum expected cost per unit time occurs at the end points of the interval \( L_i \in [L_{i,r}, L_{i,r-1}], \forall i \).

In order to examine the effect of \( m \) on the joint total expected cost per unit time, temporarily relax the integer restriction on \( m \) and take the first and second partial derivatives of \( JT \) with respect to \( m \) and obtain

\[
\frac{\partial JT}{\partial m} (Q,k_1,k_2,\ldots,k_n,L_1,L_2,\ldots,L_n,A_v,m) = -\frac{D}{Q} \left( \frac{A_v + R(A_v)}{m^2} \right) + \frac{Q}{2} h_v C_v \left(1 - \frac{D}{p}\right)
\]

and

\[
\frac{\partial^2 JT}{\partial m^2} (Q,k_1,k_2,\ldots,k_n,L_1,L_2,\ldots,L_n,A_v,m) = \frac{2D}{Q} \left( \frac{A_v + R(A_v)}{m^3} \right) > 0
\]
Therefore, \( JTEC(Q, k_1, k_2, ..., k_N, L_1, L_2, ..., L_N, A_v, m) \) is convex in \( m \) for fixed \( Q \) and \( L_i \in [L_{i,r}, L_{i,r-1}], \forall i \). As a result, the necessary condition for \( m \) to be optimal are \( JTEC(m^*) \leq JTEC(m^* - 1) \) dan are \( JTEC(m^*) \leq JTEC(m^* + 1) \).

On the other hand, by setting (4.1) equal to zero, we obtain

\[
Q = \left[ 2D \left( \frac{A_{yj} + R(A_{yj})}{m} \right) + \sum_{i=1}^{N} \left( A_i + C_i(L_i) \right) + (Kf + c \sum_{g=1}^{K} l_g) \right]^{1/2} \left[ \frac{1}{2} \sum_{i=1}^{N} h_i C_u D_i + h_u C_u \right] m \left( 1 - \frac{D}{p} \right) - 1 + \frac{2D}{p} \right], \quad L_i \in [L_{i,r}, L_{i,r-1}]. \tag{4.3}
\]

Thus for fixed \( L_i \in [L_{i,r}, L_{i,r-1}], \forall i, A_v, \) and \( m \), if \( Q \) of equation (4.3) satisfies the SLC from equation (3.5) on all buyers for their zero safety factor, then this \( Q \) and \( k_i = 0, \forall i \) is optimal and the SLC of all buyers are inactive.

On the other hand for fixed \( m, A_v, j \) and \( L_i \in [L_{i,r}, L_{i,r-1}], \forall i \), if \( Q \) from equation (4.1) doesn’t satisfy at least one of the buyer’s SLC in equation (3.2) for safety factors as zero, then \( Q \) is not optimal and SLC on one or more buyer will turn out active. Therefore for fixed \( m \) and \( L_i \in [L_{i,r}, L_{i,r-1}], \forall i \) to find optimal \( Q \) and safety factor of all the buyers with active SLC will be using Langrange multiplier technique. The Langrangian function of the joint total expected cost function is written

\[
F = \frac{D}{Q} \left[ \frac{A_{yj} + R(A_{yj})}{m} \right] + \sum_{i=1}^{N} \left( A_i + C_i(L_i) \right) + (Kf + c \sum_{g=1}^{K} l_g) \right] + \sum_{i=1}^{N} h_i C_i D_i \left[ \frac{D_i Q}{2D} + (1 - \beta_i) \sigma_i \sqrt{L_i \psi(k_i)} \right] + \frac{q}{2} h_u C_u \left[ m \left( 1 - \frac{D}{p} \right) - 1 + \frac{2D}{p} \right] + \sum_{u \in U} \lambda_u \left[ D \sigma_u \sqrt{L_u \psi(k_u)} - \alpha_u D_u Q \right] \tag{4.4}
\]

where \( \lambda_u \) is Langrange multiplier associated with buyer \( u \) (buyer with active SLC). The set of buyers with active SLC can be checked by SLC of each buyer one by one for the safety factor defined as zero and \( Q \) from equation (4.3). Let \( u \in U \) and \( w \in W \) denote set of buyers with active SLC and set of buyers with inactive SLC, respectively. If buyer \( i, i = 1, 2, ..., N, \) follows \( D \sigma_u \sqrt{L_u \psi(k_i)} / D_i Q > \forall i, \) for \( k_i = 0 \) then \( i \in U \) otherwise \( i \in W \). For fixed \( m \) and \( L_i \in [L_{i,r}, L_{i,r-1}], \forall i \) to obtained optimal solution of \( Q \) and \( k_u \) for buyers with active SLC with solution to the set equation

\[
\frac{\partial F}{\partial Q} = 0, \frac{\partial F}{\partial k_u} = 0, \frac{\partial F}{\partial \lambda_u} = 0, \forall u
\]

Thus,

\[
\frac{\partial F}{\partial Q} = \frac{1}{Q^2} \left[ \frac{A_{yj} + R(A_{yj})}{m} \right] + \sum_{i=1}^{N} \left( A_i + C_i(L_i) \right) + (Kf + c \sum_{g=1}^{K} l_g) \right] + \frac{1}{2D} \sum_{i=1}^{N} h_i C_i D_i + \frac{h_u C_u}{2} \left[ m \left( 1 - \frac{D}{p} \right) - 1 + \frac{2D}{p} \right] - \sum_{u \in U} \lambda_u \alpha_u D_u = 0 \tag{4.5}
\]

\[
\frac{\partial F}{\partial k_u} = h_u C_u \sigma_u \sqrt{L_u} - D \lambda_u \alpha_u \sqrt{L_u} \left[ 1 - \Phi(k_u) \right] = 0, \forall u \tag{4.6}
\]

\[
\frac{\partial F}{\partial \lambda_u} = D \lambda_u \sqrt{L_u \psi(k_u)} - D \lambda_u Q = 0, \forall u \tag{4.7}
\]

Now, putting \( \lambda_u \) and \( \alpha_u \) from (4.6) and (4.7), respectively, into (4.5), we get
Thus, by solving equation (4.7) we get

\[
Q = \frac{\prod_{u \in \Omega(k_u)} \psi(k_u)}{\frac{\sum_{u \in \Omega(k_u)} C_{bu} \sqrt{\sum u} + 2D \nu C_v \left[ m \left( 1 - \frac{D}{p} \right) - 1 + \frac{2D}{p} \right]}{\sum_{u \in \Omega(k_u)} C_{bu} \sqrt{\sum_u} \sum u} + \sum_{u \in \Omega(k_u)} C_{bu} \sqrt{\sum u}}}
\]

From (4.7), we have

\[
\psi(k_u) = \frac{\alpha_u D_{u} \nu Q}{\sigma_u \sqrt{\sum u}}. \quad \forall u.
\]  

The optimal solution of Q and k_u can be obtained by taking initial value of Q from equation (4.3) then solving equation (4.8) and (4.9) iteratively until convergence, for fixed and \( L_i \in [L_i, L_{i-1}], \forall i, A_v \) and m. From procedures described above optimal value of \( Q^* \) and \( k_i^* \) can be obtained for fixed routes, \( L_i \in [L_i, L_{i-1}], \forall i, \) and m. Based on the convexity and concavity behavior of objective function with respect to the decision variables the Algorithm 4.1 is developed to find the global optimal solution for the safety factor, order quantity, and lead time of each buyer, and setup cost and number of deliveries in one production cycle for known vehicle routes.

**Algorithm 4.1** Solving the integrated inventory problem for known vehicle routes.
1. Set \( JT^* = \infty \) and shipment frequency \( m = 1 \).
2. For each buyer \( i = 1, 2, ..., N \) perform 3
3. For each \( j = 0, 1, 2, 3, ..., u \) perform 3.1 to 3.4
   3.1 Set lead time of buyer \( i \) to \( L_i, \forall r = 0, 1, ..., n_i \) resulting from crashing of first components and perform step 3.2 to step 3.4 By keeping lead time fixed at \( L_{k,0} \) \( \forall k \in \{1, 2, ..., N\}\) for all other buyers.
   3.2 Compute shipment lot size \( Q^{i,r,j} \) using equation (4.3).
   3.3 If \( Q^{i,r,j} \) satisfies the SLC from equation (3.5) of all buyers for safety factor \( k_i^{i,r,j} = 0, \forall s \in \{1, 2, ..., N\} \), then set \( k_i^{i,r,j} = 0, \forall s \) and go to step 3.4
   Otherwise follow step 3.3.1 to step 3.3.5 to find shipment lot size \( Q^{i,r,j} \) and safety factor \( k_i^{i,r,j} \) of the buyers with active SLC.
   3.3.1 Identify the buyers with active SLC as follows. If the buyer \( s \in \{1, 2, ..., N\} \), \( (D \sigma_s \sqrt{L_s} \psi(k_s) / D_s Q) > \alpha_s \), then \( s \in U \),
3.3.2 Start with $Q_1^{i,r,j} = Q^{i,r,j}.$

3.3.3 Substitute $Q_1^{i,r,j}$ into equation (4.9) to evaluate $\psi(k_{u,1}^{i,r,j}), \forall u \in U$ then find $k_{u,1}^{i,r,j}$ by checking the standard loss table function table and hence $\Phi(k_{u,1}^{i,r,j})$ from the standard normal table.

3.3.4 Utilize $\psi(k_{u,1}^{i,r,j})$ and $\Phi(k_{u,1}^{i,r,j}), \forall u$ to determine $Q_2^{i,r,j}$ from equation (4.8).

3.3.5 Repeat 3.3.3 and 3.3.4 by setting $Q_1^{i,r,j} = Q_2^{i,r,j}$ and $k_{u,2}^{i,r,j} = k_{u,1}^{i,r,j}, \forall u$ until no change occurs in the values of $Q^{i,r,j}$ and $k_{u}^{i,r,j}\forall u$.

3.4 Compute the corresponding joint total expected cost $J_TEC(Q^{i,r,j}, k_1^{i,r,j}, k_2^{i,r,j}, k_3^{i,r,j}, ..., k_N^{i,r,j}, L_{1,0}, ..., L_{l,r}, ..., L_{N,0}, A_{v,j}, m)$ using equation (3.4).

3.5 Find

$$\min_{r=0,1,...,n_1} \{J_TEC(Q^{i,r,j}, k_1^{i,r,j}, k_2^{i,r,j}, k_3^{i,r,j}, ..., k_N^{i,r,j}, L_{1,0}, ..., L_{l,r}, ..., L_{N,0}, A_{v,j}, m)\}.$$ Let

$$J_TEC(Q(m), k_1(m), k_2(m), ..., k_N(m), L_{1(m)}, L_{2(m)}, ..., L_{N(m)}A_{v,j}(m), m) =$$

$$\min_{r=0,1,...,n_1} \{J_TEC(Q^{i,r,j}, k_1^{i,r,j}, k_2^{i,r,j}, k_3^{i,r,j}, ..., k_N^{i,r,j}, L_{1,0}, ..., L_{l,r}, ..., L_{N,0}, A_{v,j}, m)\}$$

then $L_{i(m)}^*$ is the optimal lead time for buyer $i$ for fixed $m$.

3.6 Find

$$\min_{j=0,1,...,u} \{J_TEC(Q^{i,r,j}, k_1, k_2, ..., k_N, L_{1}, L_{2}, ..., L_{N}, A_{v,j}, m)\}.$$ Let

$$J_TEC(Q(m), k_1(m), k_2(m), ..., k_N(m), L_{1(m)}, L_{2(m)}, ..., L_{N(m)}A_{v,j}(m), m) =$$

$$\min_{j=0,1,...,u} \{J_TEC(Q^{i,r,j}, k_1, k_2, ..., k_N, L_{1}, L_{2}, ..., L_{N}, A_{v,j}, m)\}$$ then $A_{v}^*(m)$

3.7 Take the optimal lead time of all buyers and optimal setup cost for fixed $m$ as $L_{i(m)}^*, \forall u, A_{v}^*(m)$ and follow solution procedure described in 3.2 and 3.3 to find $Q_{i}^{(m)}, k_{1}^{(m)}, k_{2}^{(m)}, ..., k_{N}^{(m)}$ and corresponding

$$J_TEC(Q_{i}^{(m)}, k_{1}^{(m)}, k_{2}^{(m)}, ..., k_{N}^{(m)}, L_{1}^{(m)}, L_{2}^{(m)}, ..., L_{N}^{(m)}, A_{v}^{(m)}, m^*)$$

is the optimal solution for fixed $m$.

3.8 If

$$J_TEC(Q_{i}^{(m)}, k_{1}^{(m)}, k_{2}^{(m)}, ..., k_{N}^{(m)}, L_{1}^{(m)}, L_{2}^{(m)}, ..., L_{N}^{(m)}, A_{v}^{(m)}, m^*) \leq$$

$$J_TEC^*,\text{ then set } J_TEC^* = J_TEC(Q_{i}^{(m)}, k_{1}^{(m)}, k_{2}^{(m)}, ..., k_{N}^{(m)}, L_{1}^{(m)}, L_{2}^{(m)}, ..., L_{N}^{(m)}, A_{v}^{(m)}, m^*), m = m + 1$$ and repeat step 2.2 to 2.4, otherwise go to step 2.6.

3.9 Set
\[ JTEC(Q(m), k_1^*(m), k_2^*(m), \ldots, k_m^*(m), L_1^*(m), L_2^*(m), \ldots, L_m^*(m), A_v^*(m), m^*) = \]
\[ JTEC(Q(m-1), k_1^{(m-1)}, k_2^{(m-1)}, \ldots, k_m^{(m-1)}, L_1^{(m-1)}, L_2^{(m-1)}, \ldots, L_m^{(m-1)}, A_v^*(m-1), (m - 1)^*) \]
then

\[ JTEC(Q(m), k_1^*(m), k_2^*(m), \ldots, k_m^*(m), L_1^*(m), L_2^*(m), \ldots, L_m^*(m), A_v^*(m), m^*) \]
is the optimal solution.

3.10 Determine the optimal order quantity for each buyer using the relationship
\[ Q_i^* = D_i Q^* / D, \forall i, \]
then
\[ (Q_i^*, k_1^*(m), k_2^*(m), \ldots, k_m^*(m), L_1^*(m), L_2^*(m), \ldots, L_m^*(m), A_v^*(m), m^*) \] gives
the optimal solution for order quantity, safety factor, lead time of each buyer, and the number of deliveries in one production cycle to all buyers.

4.2 Solving Vehicle Routing Problem

The optimal value of \( Q \) is used as input to solve vehicle routing problem (VRP) using VRP solver which based on Clark-Wright Algorithm. The solver takes input as distance matrix between each pair of location, order quantity of buyers, and capacity vehicle. As result, set of routes found, each route will be bidirectional. The vehicle dispatches from vendor to a set of buyers on a route alongside clockwise (CW) and counterclockwise (CCW), by comparing transportation time between CW and CCW routes, route with less transportation time chosen as the route for the next iteration.

4.3 Solving Integrated Inventory Model with Transportation

To solve the integrated inventory problem with transportation cost, the following algorithm has been proposed to obtain the optimal solution.

**Algorithm 4.2**

1. Assume direct shipment between vendor an the buyers. This give as many route as the number of buyers. \( R_g: 0 \rightarrow 1 \rightarrow 0, g = i = 1, 2, \ldots, N \).
2. Using the routes information find the number of routes, length of each route and transportation time from the vendor to each buyer. Solve the integrated inventory problem with help of Algorithm 4.2. This gives the order quantity of each buyer as one of the outputs.
3. Take the optimal order quantity of the buyers as input for the VRP solver and run the VRP solver to obtain set of routes.
4. Find the cumulative travel distances from vendor to all buyers along each CW route and CCW route.
5. For each route, select CW or CCW sequence of route along which cumulative travel distance from the vendor to all buyers in the sequence has smaller value.
6. Repeat Step 2 – Step 5 until the outputs of the integrated inventory problem and the VRP get stabilized.
5. Numerical Example

The example below taken from Jha and Shanker [5] and Jaggi and Arneja [6]. Here are data for vendor. \( P = 50000 \) unit per year, \( C_v = $15 \) per unit and \( h_v = 0.2 \). Data for buyers are written in Table 1, Table 2 is Lead time components, Table 3 is Setup Cost components and Table 4 is Distance matrix. The vehicle available has the following characteristics: \( f = $200 \) per vehicle, \( C = $0.1 \) per mile, \( b = 1700 \) units and \( S = 120 \) miles per day.

To solve the numerical example, we follow Algorithm 4.2 and the results are summarized in Table 4. First, assume direct shipment between vendor and buyers, which gives total number of routes as many as number of buyers. Taking the number of routes, total routes length, the transportation time for buyers. Taking the number of routes, total length route, the transportation time for buyers the other parameters as input for integrated inventory model. Algorithm 4.1 is used to find the optimal decision variables of the integrated inventory model. One of the outputs is order quantity of each buyers, which are the inputs to VRP solver. The set of CW routes obtained are 0-1-4-5-0 and 0-2-3-0. The transportation time, for example buyer 3 which lies on route 0-2-3-0 will be the sum of transportation time from vendor 0 to buyer 2, from buyer 2 to buyer 3. Therefore, the total transportation time for all buyers along route 0-2-3-0 can be calculated as the sum of transportation time for buyer 2 and buyer 3, i.e. 2.27 days. Similarly the total transportation time for the same buyers along route 0-3-2-0 which is designed as CCW route is 2.17 days. It can be observed that the transportation time for buyers along CCW route has less value than on the CW route, so CCW route 0-3-2-0 is selected between CW and CCW route. Similarly, other routes can be selected from the remaining pairs of CW and CCW routes. Next, using the routes information, the integrated inventory model is solved and get revised optimal value of decision variables. The process continue until no changing occurs on the routes information and optimal value of decision variables. Subsequent iterations performed and can be seen from Table 4 that the routes information and optimal value of decision variables get stabilized at the third iteration. Figure 2. is outputs from VRP solver from first iteration, second iteration and the third iteration. Thus, The optimal solution which are obtained at the third iteration can be read off from Table 4 as: Routes =\{0-1-3-2-0, 0-4-5-0\}, Lead time =\{22.75, 23.18, 22.99, 22.90, 23.02\}, Safety factor =\{0.255525, -
0.022905, 1.026858, 1.0461561, 0.259723, order quantity = {237.164, 1185.82, 189.731, 474.329, 237.164}, shipment frequency = 2, setup cost = 2100, and JTEC = 17003.8

Table 1. Data for buyers

<table>
<thead>
<tr>
<th>Buyers</th>
<th>$D_i$ (unit per year)</th>
<th>$\sigma_i$ (unit per day)</th>
<th>$A_i$ ($/ per order$)</th>
<th>$1 - \alpha_i$ (%)</th>
<th>$C_{bi}$ ($/ per unit$)</th>
<th>$h_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,5</td>
<td>1000</td>
<td>7</td>
<td>20</td>
<td>96</td>
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<td>0.2</td>
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<tr>
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<td>18</td>
<td>30</td>
<td>97</td>
<td>20</td>
<td>0.2</td>
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<td>800</td>
<td>10</td>
<td>25</td>
<td>98</td>
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<tr>
<td>4</td>
<td>2000</td>
<td>13</td>
<td>15</td>
<td>99</td>
<td>28</td>
<td>0.2</td>
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</table>

Table 2. Data for lead time component

<table>
<thead>
<tr>
<th>Lead time component $r$</th>
<th>Normal duration $b_{i,r}$ (days)</th>
<th>Minimum duration $a_{i,r}$ (days)</th>
<th>Unit crashing cost $C_{i,r}$ ($/day$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>4</td>
<td>0.1</td>
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Table 3. Data for setup cost component

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<tr>
<th>Setup cost Component $j$</th>
<th>Normal cost $e_i$</th>
<th>Minimum cost $d_i$</th>
<th>Crashing Setup Cost $f_i$</th>
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<tr>
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<td>900</td>
<td>400</td>
<td>3500</td>
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</table>

Table 4. Distance matrix

<table>
<thead>
<tr>
<th>Vendor</th>
<th>Buyer 1</th>
<th>Buyer 2</th>
<th>Buyer 3</th>
<th>Buyer 4</th>
<th>Buyer 5</th>
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<tbody>
<tr>
<td>Vendor</td>
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<td>109</td>
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<td>39</td>
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<td>37</td>
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<td>23</td>
<td>57</td>
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<td>23</td>
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<td>Buyer 5</td>
<td>112</td>
<td>50</td>
<td>69</td>
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Figure 2. Output from first iteration (up), output from second iteration (center), output from third iteration (below).
Table 5. Summary of solution procedure

<table>
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<tr>
<th>Iteration</th>
<th>Buyer (i)</th>
<th>( Q_i )</th>
<th>Route (Total Transportation Time along route)</th>
<th>Route with smaller transportation time (Total routes length)</th>
<th>Transportation Time ( T_i )</th>
<th>Lead Time ( L_i )</th>
<th>Safety Factor ( k_i )</th>
<th>Order Quantity ( Q_i )</th>
<th>( A_v ) ((m))</th>
<th>( JTEC )</th>
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REFERENCES


