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$H-$SUPERMAGIC LABELING ON CORONA PRODUCT OF FAN, LADDER, AND WINDMILL GRAPH WITH A PATH

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Abstract. A simple graph $G = (V, E)$ admits an H-labeling if every edge $e \in E(G)$ belongs to a subgraph of G isomorphic to H. Furthermore, G contains H-labeling if there exists a bijection function $f : V(G) \cup E(G) \rightarrow \{1, 2, \cdots, |V(G)| + |E(G)|\}$, such that for each subgraph H' of G isomorphic to H, $f(H') = \sum_{v \in V} f(v) \times \sum_{e \in E} f(e) = m(f)$ where $m(f)$ is a magic sum. Then G is an H-supermagic if $f(V) = \{1, 2, \cdots, |V(G)|\}$. This research aims to find H-super magic labeling on corona product, which: a fan graph with a path $(F_n \odot P_m)$ where $n \geq 4, m \geq 3$, a ladder graph with a path $(L_n \odot P_m)$, where $n, m \geq 3$, and a windmill graph with a path $(W_{3,m} \odot P_m)$ where $m \geq 3$. The result show that $F_n \odot P_m$ for $m \geq 3$ and $n \geq 4$ is $C_3 \odot P_m$-supermagic, a $L_n \odot P_m$ for $m, n \geq 3$ is $C_4 \odot P_m$-supermagic, and $W_{3,m} \odot P_m$ for $m \geq 3$ is $C_3 \odot P_m$-supermagic.
Keywords *: H-supermagic labeling, fan graph, ladder graph, windmill graph, path*

1. Introduction

Gallian[1] defined a graph labeling as an assignment of integers to the vertices or edges, or both, subject to certain conditions. Magic labeling was first introduced by Sedláček [3] in 1963.

The concept of H-magic graphs was introduced by Gutiérrez and Lladó [4] in 2005. Suppose $G = (V, E)$ admits an H-covering. We say that a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}$ is an H-magic labeling of G if there exists a positive integer $m(f)$, called magic sum, such that for any subgraph $H' = (V', E')$ of G isomorphics to H, the sum $\sum_{v \in V'} f(v) + \sum_{e \in E'} f(e)$ is equal to the magic sum, $m(f)$. If $f(V) = \{1, 2, \ldots, |V(G)|\}$, then we say that f is an H-supermagic labeling and $s(f)$ is a constant supermagic sum.

Lladó and Moragas [5] proved that C_3-supermagic labeling on a wheel graph W_n for $n \geq 5$ odd and a C_4-supermagic labeling of a prism graph and a book graph. Roswitha et al. [7] proved H-supermagic labeling for some classes of graphs such as a Jahangir graph, a wheel graph for even n, and a complete bipartite graph $K_{m,n}$ for $m = 2$. In this paper we found that a corona product of fan graph with a path $(F_n \odot P_m)$ is a $C_3 \odot P_m$-supermagic for $n \geq 4, m \geq 3$, a corona product of a ladder graph with a path $(L_n \odot P_m)$ is a $C_4 \odot P_m$-supermagic for $m, n \geq 3$, and a corona product of windmill graph with a path $(W_{3,m} \odot P_m)$ is a $C_3 \odot P_m$-supermagic for $m \geq 3$.

2. Main Results

2.1. **k-balanced multiset.** In [6] had introduced a technique of partitioning a multiset, called k-balanced multiset.

Let $k \in N$ and Y be a multiset that contains positive integers. Y is said to be k-balanced if there exist k subsets of Y, say $Y_1, Y_2, \ldots, Y_k$, such that for every $i \in [1, k]$, $|Y_i| = |Y_k|, \Sigma Y_i = \frac{\Sigma Y}{k} \in N$, dan $\uplus_{i=1}^{k} Y_i = Y$. If this case for every $i \in [1, k]$ then Y_i is called a balanced subset of Y.

Lemma 2.1. (Roswitha et al.[7]) *Let x and y be non negative integers. Let $X = [x + 1, x + k]$ with $|X| = k$ and $Y = [y + 1, y + k]$ where $|Y| = k$. Then the multiset $K = X \biguplus Y$ is k-balanced for $j \in [1, k]$.*

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Lemma 2.2. (Maryati [6]) *Let x, y and z be positive integers. Then multiset $Y = [x+1, x+k] \uplus [y+1, y+k] \uplus [z+1, z+k]$ is k-balanced for $k \geq 3$ odd.*

Lemma 2.3. (Maryati [6]) *Let x, y and k be integers, such that $1 \leq x \leq y$ and $k > 1$. If $X = [x, y]$ and $|X|$ is a multiple $2k$, then X is k-balanced.*

2.2. **(k,δ)-anti balanced multiset.** Inayah [2] also introduced (k,δ)-anti balanced as follows. Let $k \in N$ and X be a multiset containing positive integers. Then X is said to be (k,δ)-anti balanced if there exists k subsets of X, say $X_1, X_2, \cdots, X_k$, such that for every $i \in [1,k], |X_i| = |X_k|, \Sigma X_i = \frac{\Sigma X}{k} \in N, \uplus_{i=1}^{k} X_i = X$ and for $i \in [1, k-1], \Sigma X_{i+1} - \Sigma X_i = \delta$ is satisfied.

Lemma 2.4. Let x, y and z be non negative integers. Let $Y = [x+1, x+k] \uplus [y+1, y+k] \uplus [z+1, z+k]$ is $(k,1)$-anti balanced for $k \geq 4$ even.

Proof. For even $k \geq 4$ and every $i \in [1,k]$, we define the multisets $Y_i = \{a_1, b_i, c_i\}$, where

$$\begin{aligned} a_i &= x+i \\ b_i &= y+k+1-i \\ c_i &= z+i \end{aligned}$$

Furthermore, we define

$$\begin{aligned} A &= \{a_i | 1 \leq i \leq k\} &= [x+1, x+k] \\ B &= \{b_i | 1 \leq i \leq k\} &= [y+1, y+k] \\ C &= \{c_i | 1 \leq i \leq k\} &= [z+1, z+k]. \end{aligned}$$

For every $i \in [1,k]$ we obtain $|Y_i| = 3$; $Y_i \subset Y$ and $\uplus_{i=1}^{k} Y_i = Y$ where $k \geq 4$ even, because $\Sigma Y_i = x+y+z+k+1+i$, so for every $i \in [1,k]$, $\Sigma(Y_{i+1}) - \Sigma(Y_i) = 1$. Hence, for every $i \in [1,k]$, Y is $(k,1)$-anti balanced. □

Lemma 2.5. Let x and y be non negative integers, then multiset $Y = [x+1, x+k] \uplus [y+1, y+k]$ is $(k,1)$-anti balanced for $k \geq 3$ odd.

Proof. For odd $k \geq 3$ and every $i \in [1,k]$, we define the multisets $Y_i = \{a_1, b_i, c_i\}$, where

$$a_i = i+1,$$

$$b_i = \begin{cases} 2k+1-\lfloor\frac{k}{2}\rfloor - \lfloor\frac{i+1}{2}\rfloor, & \text{for } i \text{ even; } i \in [1,k] \\ 2k+2-\lceil\frac{i}{2}\rceil, & \text{for } i \text{ odd; } i \in [1,k]. \end{cases}$$

Furthermore, we define

$$\begin{aligned} A &= \{a_i | 1 \leq i \leq k\} &= [x+1, x+k] \\ B &= \{b_i | 1 \leq i \leq k\} &= [y+1, y+k], \end{aligned}$$

we obtain $|Y_i| = 2$; $Y_i \subset Y$ and $\uplus_{i=1}^{k} Y_i = Y$ where $k \geq 3$ odd, because

$$\Sigma Y_i = \begin{cases} x+y+k+\lceil\frac{i}{2}\rceil, & \text{for } i \text{ odd; } i \in [1,k], \\ x+y+(\frac{k+1}{2})+\lfloor\frac{i+1}{2}\rfloor, & \text{for } i \text{ even; } i \in [1,k], \end{cases}$$

so $\Sigma(Y_{i+2}) - \Sigma(Y_i) = 1$. Hence, for every $i \in [1,k]$, Y is $(k,1)$-anti balanced. □

Lemma 2.6. Let x and k be non negative integers, then multiset $X = [1, k+1] \backslash \{\frac{k}{2}+1\} \uplus [x+1, x+k]$ is $(k,1)$-anti balanced for $k \geq 4$ even.

Proof. For odd $k \geq 4$ and every $i \in [1,k]$, we define the multisets $X_i = \{a_1, b_i, c_i\}$, where

$$a_i = \begin{cases} i & \text{for } i \in [1, \frac{k}{2}] \\ i+1 & \text{for } i \in [\frac{k}{2}+1, k] \end{cases}$$

$$b_i = \begin{cases} 2k+1-2i+3, & \text{for } i \in [1, \frac{k}{2}] \\ 3k-2i+2, & \text{for } i \in [\frac{k}{2}+1, k]. \end{cases}$$

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Furthermore, we define

$$A = \{a_i|1 \le i \le k\} = [1, k+1] \backslash \{\tfrac{k}{2}+1\},$$
$$B = \{b_i|1 \le i \le k\} = [x+1, x+k],$$

we obtain $|Y_i| = 2$; $Y_i \subset Y$ and $\uplus_{i=1}^{k} Y_i = Y$ where $k \ge 3$ odd, because

$$\Sigma X_i = \begin{cases} x+k+2-i, & \text{for } i \in [1, \frac{k}{2}], \\ 2x+k+1-i, & \text{for } i \in [\frac{k}{2}+1, k], \end{cases}$$

so $\Sigma(X_i) - \Sigma(X_{i+1}) = 1$. Hence, for every $i \in 1$ and $i \in \frac{k}{2}+1$, Y is $(k,1)$-anti balanced. □

Here we provide three examples of those lemmas as follows.

(1) Let $Y = [55, 60] \uplus [61, 66] \uplus [67, 22]$ where $x = 54, y = 60, z = 66$, and $k = 6$. According to Lemma 2.4, we have 6-subsets of Y as follows. $Y_1 = \{55, 66, 67\}, Y_2 = \{56, 65, 68\}, Y_3 = \{57, 64, 69\}, Y_4 = \{58, 63, 70\}, Y_5 = \{59, 62, 71\}$, and $Y_6 = \{60, 61, 72\}$. Hence, $\Sigma(Y_{i+1}) - \Sigma(Y_i) = 1$ for $i \in [1, 6]$ and $\Sigma Y_i = 187 + i$, then Y is $(6,1)$-anti balanced.

(2) Let $Y = [2, 4] \uplus [5, 7]$ where $x = 1, y = 4$, and $k = 3$. According to Lemma 2.5, we have 3-subsets of Y as follows. $Y_1 = \{2, 7\}, Y_2 = \{3, 5\}$, and $Y_3 = \{4, 6\}$. Hence, $\Sigma(Y_{i+2}) - \Sigma(Y_i) = 1$ for $i \in [1, 3]$ and
$$\Sigma Y_i = \begin{cases} 8 + \lceil \frac{i}{2} \rceil, & \text{if } i \text{ odd, for } i \in [1, 3] \\ 7 + \lfloor \frac{i+1}{2} \rfloor, & \text{if } i \text{ even, for } i \in [1, 3], \end{cases}$$
and we have that Y is $(3,1)$-anti balanced.

(3) Let $X = [1, 5] \backslash \{3\} \uplus [6, 9]$ where $x = 5$ and $k = 4$. According Lemma 2.6, we have 4-subsets of X as follows. $X_1 = \{1, 9\}, X_2 = \{2, 7\}, X_3 = \{4, 8\}$ and $X_4 = \{5, 6\}$. Hence, $\Sigma(X_i) - \Sigma(X_{i+1}) = 1$ for $i \in 1$ and $i \in 3$. Furthermore,
$$\Sigma X_i = \begin{cases} 11 - i, & \text{for } i \in [1, 2] \\ 15 - i, & \text{for } i \in [3, 4], \end{cases}$$
then X is $(4,1)$-anti balanced.

2.3. $C_3 \odot P_m$-Supermagic Labeling On Corona Product of A Fan and A Path.

Theorem 2.1. *A graph $F_n \odot P_m$ is $C_3 \odot P_m$-supermagic for $m \ge 3, n \ge 4$.*

Proof. Let G be a graph $F_n \odot P_m$. Then $V(G) = \{v_i; 0 \le i \le n\} \uplus \{b_j; 1 \le j \le m(n+1)\}$ and $E(G) = \{v_0v_1, v_0v_2, \cdots, v_{n-1}v_n\} \uplus \{c_j; 1 \le j \le m(n+1)\} \uplus \{a_j^i; 1 \le i \le n+1; 1 \le j \le m-1\}$ where $|V(G)| = (n+1)+m(n+1)$ and $|E(G)| = 2m(n+1)n-2$. We have bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, 3m(n+1)+2n-1\}$. Given a set of labels for all vertices and edges of G denoted by Z, where $Z = [1, 3m(n+1)+2n-1]$. Partition Z into 5 sets : $A = [1, n+1], B = [n+2, (n+1)+m(n+1)], C = [(n+1)+m(n+1)+1, (2m+1)(n+1)], D = [(2m+1)(n+1)+1, 3nm+3m], E = [3mn+3m+1, 3mn+3n+2n-1]$.

All vertices and edges of subgraph H_i labeled with some certainty. The set A is used to label vertices v_i where $0 \le i \le n$, E is used to label edges $\{v_0v_1, \cdots, v_0v_n, \cdots, v_{n-1}v_n, v_nv_1\}$.

Label each vertex of F_n graph other than central vertex by

$$f(v_i) = \begin{cases} \frac{1}{2}(i+1), & \text{for } i \text{ odd; } i \in [1, n], \\ \lfloor \frac{1}{2}(n+i+1) \rfloor, & \text{for } i \text{ even; } i \in [1, n], \end{cases}$$
$$f(v_0v_i) = 3mn+3m+n+1-i, \text{ for } i \in [1, n],$$
$$f(v_iv_{i+1}) = 3mn+3m+n+i, \text{ for } i \in [1, n-1],$$

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if we set, $f(v_0) = n+1$, then the constant supermagic sum of C_3 is $f(C_3) = 9mn + 9m + 4n + 2 + \lfloor \frac{1}{2}(n+3) \rfloor$.

Furthermore, the set B is used to label b_j where $1 \leq j \leq m(n+1)$ and C is used to label c_j where $1 \leq j \leq m(n+1)$. Let $K = B \uplus C$ according to Lemma 2.1 with $x = n+2$ and $k = m(n+1)$, we have $m(n+1)$-balanced where $\Sigma K_i = 2(nm + n + m) + 3$.

Next, we used D to label edges a_j^i where $1 \leq i \leq n+1$ and $1 \leq j \leq m-1$. The proof we divided the proof by 2 cases, based on values of m.

Case 1. m is odd ($m \geq 3$).
According to Lemma 2.3 with $x = (2m+1)(n+1)+1, y = 3mn+3m$ and $|D| = m(n+1) - n - 1$, we have $(n+1)$-balanced where $\Sigma D_i = \frac{1}{2}(m-1)(5m(n+1)+n+2)$.

Case 2. m is even ($m \geq 4$).
Partition D into 2 subsets : $D_1 = [(2m+1)(n+1)+1, 2m(n+1)+5n], D_2 = [2m(n+1)+5n+1, 3mn+3m]$. For $m = 4$ defined $D = D_1$. Let $D = D_1 \uplus D_2$ for $m \geq 6$. According to Lemma 2.2 with $x = 2mn+2m+n+1, y = 2mn+2m+2n+2, z = 2mn+2m+3n+3$, we obtain $(n+1)$-balanced where $\Sigma D_{1i} = 6(mn+m+n+1)+(n+1)+\lfloor \frac{n+1}{2} \rfloor + 1$. Then based on Lemma 2.3 for $x = 2m(n+1)+5n, y = 3mn+3m$ and $|D_2| = (n+1)(m-4)$, we have $(n+1)$-balanced where $\Sigma D_{2i} = \frac{1}{2}(m-4)(5m^2n+5m+5n)$. Then $\Sigma D_i = \frac{1}{2}(5m^2n+5m^2-3mn) - 4m - 3n + 9 + \lfloor \frac{n+1}{2} \rfloor$.

The constant supermagic sum of subgraph $C_3 \odot P_m$ as follows.

$$f(C_3 \odot P_m) = \begin{cases} \frac{1}{2}(27m^2n + 27m^2 + 9m - 3n) - 3, & \text{for } m \text{ is odd; } m \geq 3, \\ 6m^2n + 6m^2 + 24mn + 27m + 21n + 24 + 3\lfloor \frac{n+1}{2} \rfloor, & \text{for } m = 4, \\ \frac{1}{2}(15m^2n + 15m^2 - 9mn) - 12m - 9n + 27 + 3\lfloor \frac{n+1}{2} \rfloor, & \text{for } m \text{ is even; } m \geq 6. \end{cases}$$

□

Figure 1, illustrates an example of $C_3 \odot P_3$-supermagic labeling on a $F_4 \odot P_3$ graph.

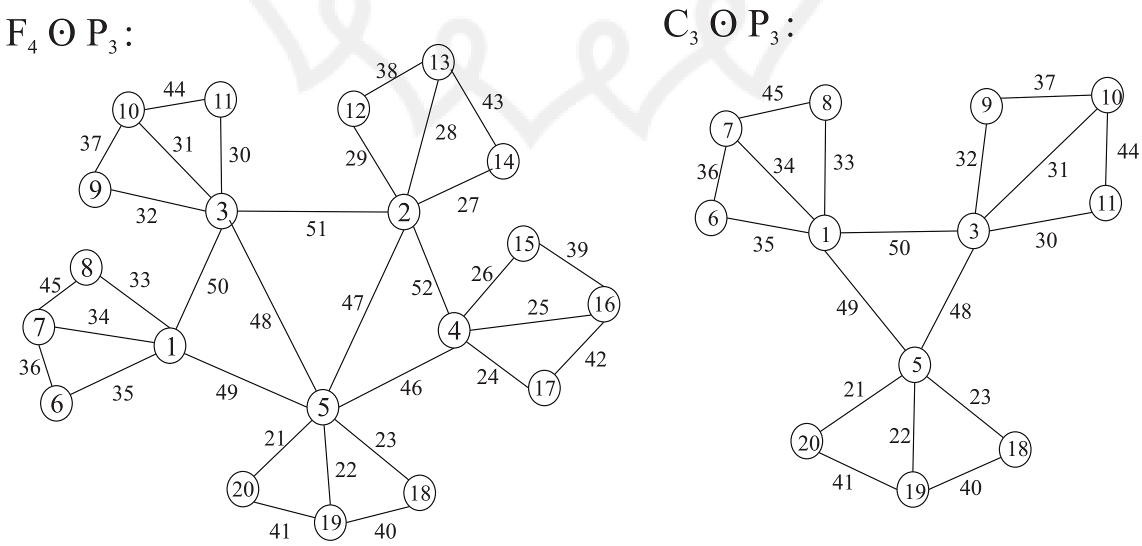

Figure 1. $C_3 \odot P_3$-supermagic labeling on a $F_4 \odot P_3$ graph.

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2.4. $C_4 \odot P_m$-Supermagic Labeling On A Ladder Corona with A Path $L_n \odot P_m$.

Theorem 2.2. *A graph $L_n \odot P_m$ is $C_4 \odot P_m$-supermagic for $m, n \geq 3$.*

Proof. Let G be a graph $L_n \odot P_m$. Let $V(G) = \{u_i; 1 \leq i \leq n\} \uplus \{v_i; 1 \leq i \leq n\} \uplus \{b_j; 1 \leq j \leq 2nm\}$ and $E(G) = \{u_1u_2, \cdots, u_{n-1}u_n, v_1v_2, \cdots, v_{n-1}v_n, u_1v_1, \cdots, u_nv_n\} \uplus \{c_j; 1 \leq j \leq 2nm\} \uplus \{a^i_j; 1 \leq i \leq 2n; 1 \leq j \leq m-1\}$ where $|V(G)| = 2n(m+1)$ and $|E(G)| = 4nm + n - 2$. We have a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, 3n(2m+1) - 2\}$. Given a set of label for all vertices and edges of G denoted by Z, where $Z = [1, 3n(2m+1) - 2]$. Partition Z into 5 sets : $A = [1, 2n], B = [2n+1, 2nm+2n], C = [2nm+2n+1, 4nm+2n], D = [4nm+2n+1, 6nm], E = [6nm+1, 6nm+3n-2]$.

All vertices and edges for each subgraph H_i labeled with some certainty. The set A is used to label vertices v_i where $1 \leq i \leq n$, E is used to label edges $\{u_1u_2, \cdots, u_{n-1}u_n, v_1v_2, \cdots, v_{n-1}v_n, u_1v_1, \cdots, u_nv_n\}$.

Label each vertex and edges of L_n graph by.

$$f(x) = \begin{cases} \text{i}, & \text{if } x = u_i; \text{ for } i \in [1, n], \\ \text{2n+1-i}, & \text{if } x = v_i; \text{ for } i \in [1, n], \\ \text{6nm+3i}, & \text{if } x = u_iu_{i+1}; \text{ for } i \in [1, n-1], \\ \text{6nm+3i-1}, & \text{if } x = v_iv_{i+1}; \text{ for } i \in [1, n-1], \\ \text{7nm-3m+10-3i}, & \text{if } x = u_iv_i; \text{ for } i \in [1, n]. \end{cases}$$

Hence, the constant supermagic sum of C_4 is $f(C_4) = 26nm + 4n - 6m + 18$.

Furthermore, the set B is used to label b_j where $1 \leq j \leq 2nm$ and C is used to label c_j where $1 \leq j \leq 2nm$. Let $K = B \uplus C$, according to Lemma 2.1 with $x = 2n$ and $k = 2nm$, we have $2nm$-balanced where $\Sigma K_i = 4nm + 4n + 1$.

Next, we used D to label edges a^i_j where $0 \leq i \leq 2n$ and $1 \leq j \leq m-1$. The proof is divided by 2 cases, based on values of m.

Case 1. m is odd ($m \geq 3$).
According to Lemma 2.3 with $x = 4nm + 2n + 1, y = 6nm$ and $|D| = 2n(m-1)$, we have $2n$-balanced where $\Sigma D_i = \frac{2nm-2n}{4n}(10nm + 2n + 1)$.

Case 2. m is even ($m \geq 4$).
Partition D into 2 subsets : $D_1 = [4nm + 2n + 1, 4nm + 8], D_2 = [4nm + 8n + 1, 6nm]$. For $m = 4$ defined $D = D_1$. Let $D = D_1 \uplus D_2$ for $m \geq 6$, according to Lemma 2.4 with $x = 4nm + 2n, y = 4nm + 3n + 3, z = 4nm + 3n + 3$, we obtain $(k, 1)$-anti balanced where $\Sigma D_{1i} = 12nm + 12n + 7 + i$. Then based on Lemma 2.1 for $x = 4nm + 8n, y = 6nm - 2n$ and $|D_2| = 2n(m-4)$, we have $2n$-balanced where $\Sigma D_{2i} = 2(4nm + 10n) - 1$. Then $\Sigma D_i = 20nm + 32n + 8 + i$.

The constant supermagic sum of subgraph $C_4 \odot P_m$ as follows.

$$f(C_4 \odot P_m) = \begin{cases} 16m^2n + 56nm + 8n + 4m + 4, & \text{for } m \text{ is odd; } m \geq 3 \\ 16m^2n + 90nm + 52n - 2m + 60, & \text{for } m = 4 \\ 16m^2n + 122nm + 132n - 2m + 64, & \text{for } m \text{ is even; } m \geq 6. \end{cases}$$

□

Figure 2, illustrates an example of $C_4 \odot P_3$-supermagic labeling on a $L_3 \odot P_3$ graph.

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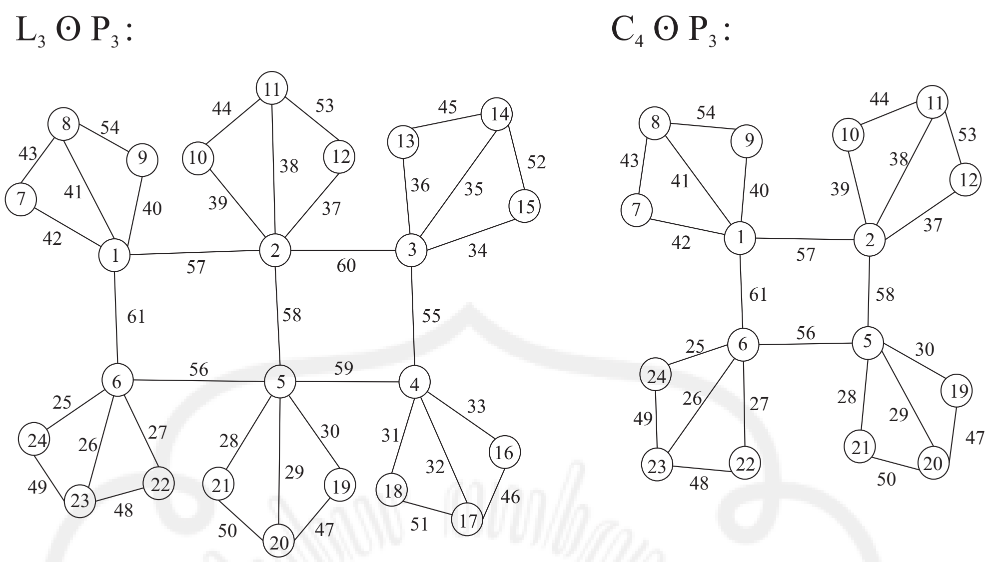

Figure 2. $C_4 \odot P_3$-supermagic labeling on a $L_3 \odot P_3$ graph.

2.5. $C_3 \odot P_m$-Supermagic Labeling On A Windmill Corona with A Path $W_{3,m} \odot P_m$.

Theorem 2.3. *A graph $W_{3,m} \odot P_m$ is $C_3 \odot P_m$-supermagic for $m \geq 3$.*

Proof. Let G be a graph $W_{3,m} \odot P_m$. Let $V(G) = \{v_i; 0 \leq i \leq 2m\} \uplus \{b_j; 1 \leq j \leq (2m+1)m\}$ and $E(G) = \{v_0v_1, \cdots, v_0v_n, v_0u_1, \cdots, v_0u_n, v_1v_1, \cdots, v_nu_n\} \uplus \{c_j; 1 \leq j \leq (2m+1)m\} \uplus \{a_j^i; 1 \leq i \leq 2m+1; 1 \leq j \leq m-1\}$ where $|V(G)| = 2m^2+3m+1$ and $|E(G)| = 6m^2$. We have a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, 8m^2+3m+1\}$. Given a set of label for all vertices and edges of G denoted by Z, where $Z = [1, 8m^2+3m+1]$. Partition Z into 5 sets : $A = [1, 2m+2], B = [2m+2, 2m^2+3m+1], C = [2m^2+3m+2, 4m^2+4m+1], D = [4m^2+4m+2, 6m^2+3m], E = [6m^2+3m+1, 6m^2+6m]$.

All vertices and edges for each subgraph H_i labeled with some certainty. The set A is used to label vertices v_i where $1 \leq i \leq m$, E is used to label edges $\{v_0v_1, v_0v_2, \cdots, v_0v_n, v_0u_1, v_0u_2, \cdots, v_0u_n, v_1v_1, v_2u_2, \cdots, v_nu_n\}$.

Case 1. m is even ($m \geq 4$).

(1) Lemma 2.6 is applied to label each vertex of $W_{3,m}$ graph other than central vertex by.

$$f(v_i) = \begin{cases} i, & \text{for } i \in [1, \frac{m}{2}], \\ i+1, & \text{for } i \in [\frac{m}{2}+1, m], \end{cases}$$

and

$$f(u_i) = \begin{cases} 2m-2i+3, & \text{for } i \in [1, \frac{m}{2}], \\ 3m-2i+2, & \text{for } i \in [\frac{m}{2}+1, m]. \end{cases}$$

(2) Label each edge of $W_{3,m}$ graph as follows.

$$f(v_0v_i) = \begin{cases} 6m^2+\frac{7}{2}m+i, & \text{for } i \in [1, \frac{m}{2}], \\ 6m^2+\frac{5}{2}m+i, & \text{for } i \in [\frac{m}{2}+1, m], \end{cases}$$

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$$f(v_0u_i) = \begin{cases} 6m^2 + \frac{9}{2}m + 1 - i, & \text{for } i \in [1, \frac{m}{2}], \\ 6m^2 + \frac{11}{2}m + 1 - i, & \text{for } i \in [\frac{m}{2} + 1, m], \end{cases}$$

and

$$f(v_iu_i) = \begin{cases} 6m^2 + \frac{11}{2}m + i, & \text{for } i \in [1, \frac{m}{2}], \\ 6m^2 + \frac{9}{2}m + i, & \text{for } i \in [\frac{m}{2} + 1, m]. \end{cases}$$

For $f(v_0) = \frac{m}{2} + 1$. Hence, the constant supermagic sum of C_3 is $f(C_3) = 18m^2 + 16m + 5$.

Case 2. m is odd ($m \geq 3$).

(1) Lemma 2.5 is applied to label every vertex of $W_{3,m}$ graph as follows.

$$f(u_i) = \begin{cases} 2m + 1 - \lfloor \frac{m}{2} \rfloor - \lfloor \frac{i+1}{2} \rfloor, & \text{for } i \text{ even, } i \in [1, m], \\ 2m + 2 - \lceil \frac{i}{2} \rceil, & \text{for } i \text{ even, } i \in [1, m], \end{cases}$$

and $f(v_i) = i + 1$; for $i \in [1, m]$.

(2) Label each edge of $W_{3,m}$ graph as follows.

$$f(v_0v_i) = \begin{cases} 6m^2 + 3m + \lceil \frac{m}{2} \rceil - \lfloor \frac{i}{2} \rfloor, & \text{if } i \text{ odd, for } i \in [1, m] \\ 6m^2 + 4m + 2 - \lceil \frac{i+1}{2} \rceil, & \text{if } i \text{ even, for } i \in [1, m], \end{cases}$$

$$f(v_0u_i) = \begin{cases} 6m^2 + 5m - \lceil \frac{m}{2} \rceil + \lceil \frac{i}{2} \rceil, & \text{if } i \text{ odd, for } i \in [1, m] \\ 6m^2 + 4m + 2 - \lfloor \frac{i+1}{2} \rfloor, & \text{if } i \text{ even, for } i \in [1, m], \end{cases}$$

and

$$f(v_iu_i) = \begin{cases} 6m^2 + 5m + \lceil \frac{m}{2} \rceil - \lfloor \frac{i}{2} \rfloor, & \text{if } i \text{ odd, for } i \in [1, m] \\ 6m^2 + 6m + 2 - \lceil \frac{i+1}{2} \rceil, & \text{if } i \text{ even, for } i \in [1, m], \end{cases}$$

Hence, the constant supermagic sum of C_3 is $f(C_3) = 18m^2 + 15m + \lceil \frac{m}{2} \rceil + 5$.

Furthermore, the set B is used to label a_j where $1 \leq j \leq ((2m+1)m)$ and C is used to label b_j where $1 \leq j \leq ((2m+1)m)$. Let $K = B \uplus C$, according to Lemma 2.1 with $x = 2m+2$ and $k = (2m+1)m$, we have $(2m+1)m$-balanced where $\Sigma K_i = 4m^2 + 6m + 3$.

Next, we used D to label edges a^i_j, where $1 \leq i \leq 2m+1$ and $1 \leq j \leq m-1$. The proof is divided by 2 cases, based on values of m.

Case 1. m is odd ($m \geq 3$).
According to Lemma 2.3 with $x = 4m^2+4m+2, y = 6m^2+3m$ and $|D| = (2m+1)(m-1)$, we have $(2m+1)$-balanced where $\Sigma D_i = \frac{1}{2}(m-1)(10m^2 + 7m + 2)$.

Case 2. m is even ($m \geq 4$).
Partition D into 2 sets : $D_1 = [4m^2+4m+2, 4m^2+10m+4], D_2 = [4m^2+10m+5, 6m^2+3m]$. For $m = 4$ defined $D = D_1$. Let $D = D_1 \uplus D_2$ for $m \geq 6$. According to Lemma 2.2 with $x = 4m^2 + 4m + 1, y = 4m^2 + 6m + 2$, and $z = 4m^2 + 8m + 3$, we obtain D_1 is $(2m+1)$-balanced where $\Sigma D_{1i} = 12m^2 + 20m + 8 + \lceil \frac{2m+1}{2} \rceil$. Then, based on Lemma 2.3 for $x = 4m^2 + 10m + 5, y = 6m^2 + 3m$ and $|D_2| = 4m + 2$, we have $(2m+1)$-balanced where $\Sigma D_{2i} = \frac{1}{2}(m-4)(10m^2 + 13m + 5)$. Then $\Sigma D_i = 5m^3 - \frac{3}{2}m^2 - \frac{7}{2}m - 2 + \lceil \frac{2m+1}{2} \rceil$.

Furthermore, the constant supermagic sum of a subgraph $C_3 \odot P_m$ is.

$$f(C_3 \odot P_m) = \begin{cases} 27m^3 + \frac{63}{2}m^2 + \frac{33}{2}m + \lceil \frac{m}{2} \rceil + 2, & \text{for } m \text{ odd }, m \geq 3 \\ 12m^3 + 72m^2 + 85m + 29 + 3\lceil \frac{2m+1}{2} \rceil, & \text{for } m = 4 \\ 27m^3 + \frac{63}{2}m^2 + \frac{29}{2}m - 1 + 3\lceil \frac{2m+1}{2} \rceil, & \text{for } m \text{ even }, m \geq 6. \end{cases}$$

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□

Figure 3 illustrates an example of $C_3 \odot P_m$-supermagic labeling on a $W_{3,3} \odot P_3$ graph.

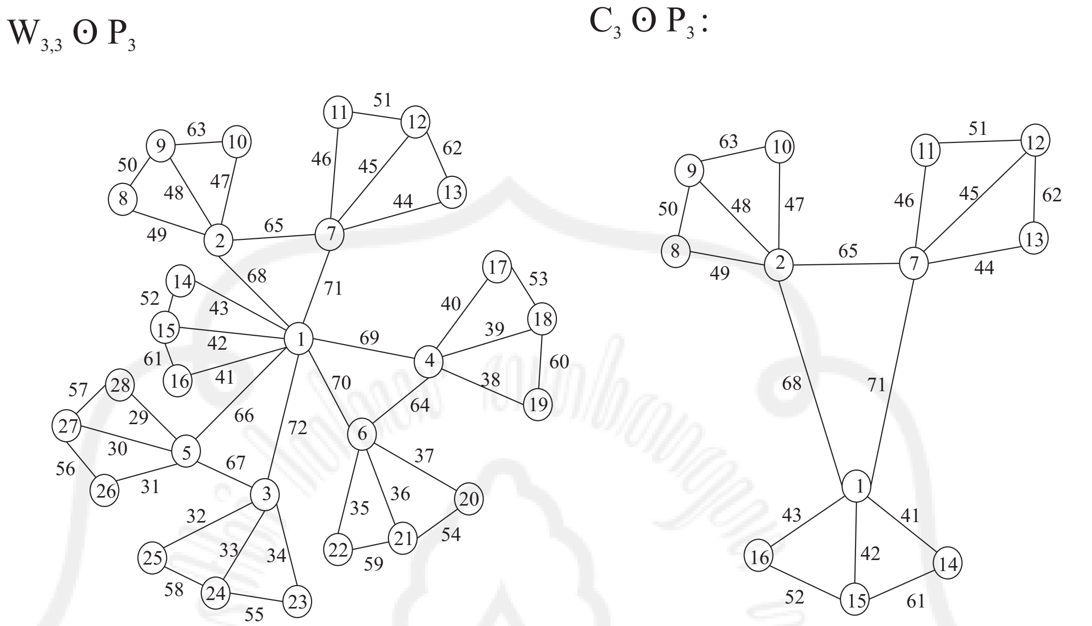

Figure 3. $C_3 \odot P_3$-supermagic labeling on a $W_{3,3} \odot P_3$ graph.

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