THE M_α −STIELTJES INTEGRAL

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ABSTRACT. Based on the McShane integral, we define the M_α −Stieltjes integral for real-valued function. In this paper, we will need the definition of an increasing function to bring the definition of the M_α integral into the M_α −Stieltjes integral. The purpose of this paper is to define the M_α −Stieltjes integral as a generalization of the M_α integral and investigate some of its basic properties such as uniqueness of the integral value, linearity, etc.

Keywords. McShane-Stieltjes integral, M_α-partition, M_α integral, and increasing function

I. Introduction and Basic Definitions

There are two types of integral, those are descriptive integral and constructive integral. Both integrals develop rapidly. McShane integral is the one of constructive integral type constructed by McShane in 1960. Some authors like Jae Myung Park [4], Ju Han Yoon [6], and Russel A. Gordon [2] say that Lebesgue integral is equivalent to McShane integral. In 1997, Yoon [6] defined McShane-Stieltjes integral and proved some of its properties. Later on, Park [4] defined M_α integral based on McShane integral, and proved some properties. From those papers, the authors are motivated to discuss the M_α −Stieltjes integral so that we can define the M_α −Stieltjes integral as a generalization of the M_α integral and observe some properties.

Definition 1.1 (Park, 2010) *Given* $\wp = \{([x_{i-1}, x_i], \xi_i), 1 \leq i \leq n\}$, *where* $([x_{i-1}, x_i], \xi_i), 1 \leq i \leq n$ *are interval-point pairs and* $[x_{i-1}, x_i], 1 \leq i \leq n$ *are non-overlapping sub-intervals of* $[a, b]$ *that is* $[x_{i-1}, x_i]^0 \cap [x_{j-1}, x_j]^0 \neq \emptyset$, *for* $i \neq j$. *Let* δ *be a positive function on* $[a, b]$.

(1) *$\wp$ is a δ-fine McShane partition of $[a, b]$ if $[x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ and $\xi_i \in [a, b]$ for all i = 1, 2, 3, …, n, and $\bigcup_{i=1}^{n}[x_{i-1}, x_i] = [a, b]$.*

(2) *$\wp$ is a δ-fine M_α partition of $[a, b]$ for a constant $\propto > 0$ if it is a δ-fine McShane partition of $[a, b]$ and satisfying the $\sum_{i=1}^{n} dist([x_{i-1}, x_i], \xi_i) < \propto$ where $dist([x_{i-1}, x_i], \xi_i) = \inf\{|t_i - \xi_i| : t_i \in [x_{i-1}, x_i]$.*

(3) a δ-fine Henstock partition of [a,b] if $\xi_i \in [x_{i-1}, x_i] \subset \left(\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)\right)$ for all i = 1, 2, 3,…, n.

Let $f: [a, b] \rightarrow R$. Given a $\delta - fine$ partition $\wp = \{([x_{i-1}, x_i], \xi_i), 1 \leq i \leq n\}$ we write $S(f, \wp) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i$ for integral sum over $\wp$, where $\Delta x_i = x_i - x_{i-1}$. Park [4] define the M_α integral as follow,

Definition 1.2 *Let α > 0 be a constant. A function $f: [a, b] \rightarrow R$ is M_α integrable if there exists a real number L such that for each ϵ > 0 there is a positive function $\delta: [a, b] \rightarrow R^+$ such that $|S(f, \wp) - L| < \epsilon$ for each δ-fine M_α partition $\wp$ = $\{([x_{i-1}, x_i], \xi_i), 1 \leq i \leq n\}$ of $[a, b]$.*

L is called the M_α integral of f on $[a, b]$. Concern to the M_α −Stieltjes integral, we need the concept of an increasing function φ. A function $\varphi: [a, b] \rightarrow R$ is called the increasing function on $[a, b]$ if $x_1, x_2 \in [a, b]$ where $x_1 < x_2$ then $f(x_1) < f(x_2)$. By taking $\varphi(x) = x$, the M_α integral is seen to be a special case of the M_α −Stieltjes integral.

II. Main Results

Based on the concept of M_α integral and McShane-Stieltjes Integral, we define the M_α −Stieltjes integral. Let $P = \{([x_{i-1}, x_i], \xi_i), 1 \leq i \leq n\}$ is a $\delta - fine$ M_α partition of $[a, b]$, then the M_α −Stieltjes integral sum of f with respect to φ over $\wp$ is defined by $S(f_\varphi, \wp) = \sum_{i=1}^{n} f(\xi_i)\Delta\varphi(x_i)$ and the M_α −Stieltjes integral of f with respect to φ on $[a, b]$ denoted by $L = (M_\alpha S) \int_a^b f \, d\varphi$.

Definition 2.1. *Let φ be an increasing function on $[a, b]$ and α > 0 be a constant. A function $f: [a, b] \rightarrow R$ is M_α −Stieltjes integrable with respect to φ if there exists a real number L with the following property : for each ϵ > 0 there is a positive function $\delta: [a, b] \rightarrow R^+$ such that $\left|S(f_\varphi, \wp) - L\right| < \epsilon$ for each δ-fine M_α partition $\wp$ = $\{([x_{i-1}, x_i], \xi_i), 1 \leq i \leq n\}$ of $[a, b]$.*

L is called the M_α −Stieltjes integral of f with respect to φ on $[a, b]$. The collection of all M_α −Stieltjes integrable functions with respect to φ on $[a, b]$ denoted by $M_\alpha S[a,b]$. We next verify the basic properties of the M_α −Stieltjes integral which contained in these following Theorems.

Theorem 2.1 *Let* $f:[a,b]\to R$ *and* φ *be an increasing function on* $[a,b]$*. If* $f \in M_\alpha S[a,b]$ *then it's integral is unique.*

Proof. Let $\epsilon > 0$. Since $f \in M_\alpha S[a,b]$, suppose that K, L are the M_α −Stieltjes integrals of f written as $(M_\alpha S)\int_a^b f\,d\varphi = K$ and $(M_\alpha S)\int_a^b f\,d\varphi = L$ where $K \neq L$. There is a positive function $\delta_1:[a,b]\to R^+$ such that for each $\delta_1 - fine$ M_α partition $\wp_1 = \{([x_{i-1},x_i],\xi_i), 1\le i\le n\}$ of $[a,b]$, we have $|S(f_\varphi,\wp_1) - K| < \epsilon/2$, and there is a positive function $\delta_2:[a,b]\to R^+$ such that for each $\delta_2 - fine$ M_α partition $\wp_2 = \{([x_{i-1},x_i],\xi_i), 1\le i\le n\}$ of $[a,b]$, we have $|S(f_\varphi,\wp_2) - L| < \epsilon/2$. We define a positive function $\delta:[a,b]\to R^+$ where $\delta(\xi) = \min\{\delta_1(\xi),\delta_2(\xi)\}$ such that for each $\delta - fine$ M_α partition satisfies :

$$|K-L| \le |K - S(f_\varphi,\wp)| + |S(f_\varphi,\wp) - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon > 0$, Theorem 2.2.9 on Bartle [1] proved that $K - L = 0$, it implies that $K = L$ which contradict to the assumption. Hence, the theorem is proved. □

Theorem 2.2 (Linear Space) *If* $f, g \in M_\alpha S[a,b]$ *and* $\lambda \in R$*, then* $(f+g) \in M_\alpha S[a,b]$ *and* $\lambda f \in M_\alpha S[a,b]$*, and satisfies:*

1) $(M_\alpha S)\int_a^b (f+g)\,d\varphi = (M_\alpha S)\int_a^b f\,d\varphi + (M_\alpha S)\int_a^b g\,d\varphi$

2) $(M_\alpha S)\int_a^b \lambda f\,d\varphi = \lambda (M_\alpha S)\int_a^b f\,d\varphi$

Proof. Let $\epsilon > 0$. (1) By hypothesis $f, g \in M_\alpha S[a,b]$, called $(M_\alpha S)\int_a^b f\,d\varphi = K$ and $(M_\alpha S)\int_a^b g\,d\varphi = L$. There is a positive function $\delta_1:[a,b]\to R^+$ such that for each $\delta_1 - fine$ M_α partition $\wp_1 = \{([x_{i-1},x_i],\xi_i), 1\le i\le n\}$ of $[a,b]$, we have $|S(f_\varphi,\wp_1) - K| < \epsilon/2$ and there is a positive function $\delta_2:[a,b]\to R^+$ such that for each $\delta_2 - fine$ M_α partition $\wp_2 = \{([x_{i-1},x_i],\xi_i), 1\le i\le n\}$ of $[a,b]$, we have $|S(g_\varphi,\wp_2) - L| < \frac{\epsilon}{2}$. Then we define a positive function $\delta:[a,b]\to R^+$ where $\delta(\xi) = \min\{\delta_1(\xi),\delta_2(\xi)\}$ such that for each $\delta - fine$ M_α partition, we have:

$|S((f+g)_\varphi,\wp) - (K+L)| = |\sum_{i=1}^{n}(f+g)(\xi_i)\Delta\varphi(x_i) - (K+L)|$

$$= \left| \sum_{i=1}^{n} (f(\xi_i)\Delta\varphi(x_i) + g(\xi_i)\Delta\varphi(x_i)) - (K + L) \right|$$

$$\leq \left| \sum_{i=1}^{n} f(\xi_i)\Delta\varphi(x_i) - K \right| + \left| \sum_{i=1}^{n} g(\xi_i)\Delta\varphi(x_i) - L \right|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

It implies that $f + g \in M_\alpha S[a, b]$ and satisfies:

$$(M_\alpha S) \int_a^b (f + g) d\varphi = K + L = (M_\alpha S) \int_a^b f \, d\varphi + (M_\alpha S) \int_a^b g \, d\varphi.$$

(2) By hypothesis $f \in M_\alpha S[a, b]$, called $(M_\alpha S) \int_a^b f \, d\varphi = K$ and $\lambda \in R$. There is a positive function $\delta: [a, b] \to R^+$ such that for each $\delta - fine\ M_\alpha$ partition $\wp = \{([x_{i-1}, x_i], \xi_i), 1 \leq i \leq n\}$ of $[a, b]$ and $|S(f_\varphi, \wp) - K| < \epsilon / |\lambda|$, we have

$$|S(\lambda f_\varphi, \wp) - \lambda K| = \left| \sum_{i=1}^{n} \lambda f(\xi_i)\Delta\varphi(x_i) - \lambda K \right| = |\lambda| \left| \sum_{i=1}^{n} f(\xi_i)\Delta\varphi(x_i) - K \right|$$

$$< |\lambda| \frac{\epsilon}{|\lambda|} = \epsilon$$

It implies that $\lambda f \in M_\alpha S[a, b]$ and satisfies:

$$(M_\alpha S) \int_a^b \lambda f \, d\varphi = \lambda K = \lambda (M_\alpha S) \int_a^b f \, d\varphi.$$

From (1) and (2), these complete the proof of the theorem. □

Theorem 2.3. *Let* $f: [a, b] \to R$. *If* $f \in M_\alpha S[a, b]$ *and* $[c, d] \subseteq [a, b]$ *then* $f \in M_\alpha S[c, d]$.

Proof. Let $\epsilon > 0$. Suppose $f \in M_\alpha S[a, b]$, there is a positive function $\delta: [a, b] \to R^+$ such that

$$|S(f_\varphi, \wp) - S(f_\varphi, Q)| < \epsilon$$

for any $\delta - fine\ M_\alpha$ partitions $\wp$ and Q of $[a, b]$. We define $\wp_1, \wp_2$ are $\delta - fine\ M_\alpha$ partitions of $[c, d]$, $\wp_3$ is $\delta - fine\ M_\alpha$ partition of $[a, c]$, and $\wp_4$ is $\delta - fine\ M_\alpha$ partition of $[d, b]$. Then $\wp_3 \cup \wp_1 \cup \wp_4$ and $\wp_3 \cup \wp_2 \cup \wp_4$ are $\delta - fine\ M_\alpha$ partitions of $[a, b]$. By using Theorem 2.4 we have:

$$|S(f_\varphi, \wp_1) - S(f_\varphi, \wp_2)| = |S(f_\varphi, \wp_3 \cup \wp_1 \cup \wp_4) - S(f_\varphi, \wp_3 \cup \wp_2 \cup \wp_4)| < \epsilon.$$

Thus, $f \in M_\alpha S[c, d]$ and the theorem is proved. □

Theorem 2.4. *Let φ be an increasing function on $[a,b]$. Let $f:[a,b]\to R$ and let $c\in[a,b]$. If f is M_α −Stieltjes integrable with respect to φ on each of the interval $[a,c]$ and $[c,b]$, then f is M_α −Stieltjes integrable with respect to φ on $[a,b]$ and*

$$(M_\alpha S)\int_a^b f\,d\varphi=(M_\alpha S)\int_a^c f\,d\varphi+(M_\alpha S)\int_c^b f\,d\varphi.$$

Proof. Let $\epsilon>0$. By hypothesis $f\in M_\alpha S[a,b]$. Since $c\in[a,b]$, suppose that f is M_α −Stieltjes integrable with respect to φ on each of the interval $[a,c]$ and $[c,b]$, called $(M_\alpha S)\int_a^c f\,d\varphi=K$ and $(M_\alpha S)\int_c^b f\,d\varphi=L$. There is a positive function $\delta_1:[a,c]\to R^+$ such that for each $\delta_1-fine\ M_\alpha$ partition $\wp_1=\{([x_{i-1},x_i],\xi_i),1\le i\le n\}$ of $[a,c]$, we have $|S(f_\varphi,\wp_1)-K|<\epsilon/2$ and there is a positive function $\delta_2:[c,b]\to R^+$ such that for each $\delta_2-fine\ M_\alpha$ partition $\wp_2=\{([x_{i-1},x_i],\xi_i),1\le i\le n\}$ of $[c,b]$, we have $|S(f_\varphi,\wp_2)-L|<\frac{\epsilon}{2}$. Then we define a positive function $\delta:[a,b]\to R^+$ where

$$\delta(\xi)=\begin{cases}\min\{\delta_1(x),c-x\}, & if\ a\le x<c\\ \min\{\delta_1(c),\delta_2(c)\}, & if\ x=c\\ \min\{\delta_2(x),x-c\}, & if\ c<x\le b\end{cases}$$

Let $\wp$ be a $\delta-fine\ M_\alpha$ partition of $[a,b]$, then $\wp=\wp_1+\wp_2$. Since, $S(f_\varphi,\wp)=S(f_\varphi,\wp_1)+S(f_\varphi,\wp_2)$, we obtain

$|S(f_\varphi,\wp)-(K+L)|=|S(f_\varphi,\wp_1)+S(f_\varphi,\wp_2)-(K+L)|$

$\le|S(f_\varphi,\wp_1)-K|+|S(f_\varphi,\wp_2)-L|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$

Hence, the function f is M_α −Stieltjes integrable with respect to φ on $[a,b]$ and

$$(M_\alpha S)\int_a^b f\,d\varphi=K+L=(M_\alpha S)\int_a^c f\,d\varphi+(M_\alpha S)\int_c^b f\,d\varphi. \qquad \square$$

Theorem 2.5. *Let φ_1,φ_2 be some increasing functions on $[a,b]$.*

1) *If f is M_α −Stieltjes integrable with respect to φ_1 and φ_2 on $[a,b]$, then f is M_α −Stieltjes integrable with respect to $(\varphi_1+\varphi_2)$ and satisfies:*

$$(M_\alpha S)\int_a^b f\,d(\varphi_1+\varphi_2)=(M_\alpha S)\int_a^b f\,d\varphi_1+(M_\alpha S)\int_a^b f\,d\varphi_2.$$

2) *If f is M_α −Stieltjes integrable with respect to φ and $\lambda\in R^+$ then f is M_α −Stieltjes integrable with respect to $\lambda\varphi$ and satisfies*

$$(M_\alpha S)\int_a^b f\,d(\lambda\varphi)=\lambda(M_\alpha S)\int_a^b f\,d\varphi.$$

Proof. Let $\epsilon > 0$. 1) By hypothesis $f \in M_\alpha S[a,b]$, called $(M_\alpha S)\int_a^b f\,d\varphi_1 = K$ and $(M_\alpha S)\int_a^b f\,d\varphi_2 = L$. There is a positive function $\delta_1: [a,b] \to R^+$ such that for each $\delta_1 - fine$ M_α partition $\wp_1 = \{([x_{i-1}, x_i], \xi_i),\ 1 \le i \le n\}$ of $[a,b]$, we have $|S(f_{\varphi_1}, \wp_1) - K| < \epsilon/2$ and there is a positive function $\delta_2: [a,b] \to R^+$ such that for each $\delta_2 - fine$ M_α partition $\wp_2 = \{([x_{i-1}, x_i], \xi_i),\ 1 \le i \le n\}$ of $[a,b]$, we have $|S(f_{\varphi_2}, \wp_2) - L| < \frac{\epsilon}{2}$. Then we define a positive function $\delta: [a,b] \to R^+$ where $\delta(\xi) = \min\{\delta_1(\xi), \delta_2(\xi)\}$ such that for each $\delta - fine$ M_α partition $\wp$ of $[a,b]$, we have:

$$|S(f_{(\varphi_1+\varphi_2)}, \wp) - (K+L)| = |\textstyle\sum_{i=1}^n f(\xi_i)\Delta(\varphi_1+\varphi_2)(x_i) - (K+L)|$$

$$= \left|\sum_{i=1}^n f(\xi_i)((\varphi_1+\varphi_2)(x_i) - (\varphi_1+\varphi_2)(x_{i-1})) - (K+L)\right|$$

$$= \left|\sum_{i=1}^n f(\xi_i)(\varphi_1(x_i) - \varphi_1(x_{i-1}) + \varphi_2(x_i) - \varphi_2(x_{i-1})) - (K+L)\right|$$

$$= \left|\sum_{i=1}^n f(\xi_i)(\Delta\varphi_1(x_i) + \Delta\varphi_2(x_i)) - (K+L)\right|$$

$$\le \left|\sum_{i=1}^n f(\xi_i)\Delta\varphi_1(x_i) - K\right| + \left|\sum_{i=1}^n f(\xi_i)\Delta\varphi_2(x_i)) - L)\right|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

It implies that $f_{(\varphi_1+\varphi_2)} \in M_\alpha S[a,b]$ and satisfies:

$$(M_\alpha S)\int_a^b f\,d(\varphi_1+\varphi_2) = K + L = (M_\alpha S)\int_a^b f\,d\varphi_1 + (M_\alpha S)\int_a^b f\,d\varphi_2.$$

2) By hypothesis $f \in M_\alpha S[a,b]$, $(M_\alpha S)\int_a^b f\,d\varphi = K$ and λ is a positive constant. There is a positive function $\delta: [a,b] \to R^+$ such that for each $\delta - fine$ M_α partition $\wp = \{([x_{i-1}, x_i], \xi_i), 1 \le i \le n\}$ of $[a,b]$ and $|S(f_{\lambda\varphi}, \wp) - K| < \epsilon/\lambda$, we have

$$|S(f_{\lambda\varphi}, \wp) - \lambda K| = \left|\sum_{i=1}^n f(\xi_i)\lambda\Delta\varphi(x_i) - \lambda K\right| = \lambda\left|\sum_{i=1}^n f(\xi_i)\Delta\varphi(x_i) - K\right|$$

$$< \lambda\frac{\epsilon}{\lambda} = \epsilon.$$

It implies that $f_{\lambda\varphi} \in M_\alpha S[a,b]$ and satisfies:

$$(M_\alpha S)\int_a^b f\, d\lambda\varphi = \lambda K = \lambda (M_\alpha S)\int_a^b f\, d\varphi.$$

From (1) and (2), the theorem is completed. □

Lemma 2.1. (Saks-Henstock Lemma) *Let* $f: [a,b] \to R$ *be* M_α *−Stieltjes integrable with respect to* φ *on* $[a,b]$. *Let* $\epsilon > 0$, *there exist a positive function* $\delta: [a,b] \to R^+$ *such that for each* $\delta - fine$ M_α *partition* $\wp = \{([x_{i-1},x_i],\xi_i), 1 \le i \le n\}$ *of* $[a,b]$, *we have*

$$\left|S(f_\varphi,\wp) - \int_a^b f\, d\varphi\right| < \epsilon.$$

Particularly, if $\wp' = \{([x_{i-1},x_i],\xi_i), 1 \le i \le m\}$ *is an arbitrary* $\delta - fine$ *partial* M_α *partition of* $[a,b]$, *we have*

$$\left|S(f_\varphi,\wp') - \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} f\, d\varphi\right| \le 2\epsilon$$

Proof. We assume $\wp' = \{([x_{i-1},x_i],\xi_i), 1 \le i \le m\}$ is an arbitrary $\delta - fine$ partial M_α partition of $[a,b]$, then the complement $[a,b] \setminus \bigcup_{i=1}^{m}[x_{i-1},x_i]$ can be expressed as a fine collection of closed subintervals and we denote $[a,b] \setminus \bigcup_{i=1}^{m}[x_{i-1},x_i] = \bigcup_{j=1}^{k}[x_{j-1},x_j]$.

Since $\epsilon > 0$ be arbitrary, there is a positive function δ_j on $[x_{j-1},x_j]$ such that if $\wp_j$ is a $\delta_j - fine$ M_α partition of $[x_{j-1},x_j]$, we have

$$\left|S(f_\varphi,\wp_j) - \int_{x_{j-1}}^{x_j} f\, d\varphi\right| < \epsilon/k.$$

Assume that $\delta_j(\xi) \le \delta(\xi)$ for all $\xi \in [a,b]$. Let $\wp_0 = \wp' \cup \wp_1 \cup \wp_2 \cup \dots \cup \wp_k$, obviously, $\wp_0$ is $\delta - fine$ M_α partition of $[a,b]$, we have

$$\left|S(f_\varphi,\wp_0) - \int_a^b f\, d\varphi\right| = \left|S(f_\varphi,\wp') + \sum_{j=1}^{k} S(f_\varphi,\wp_j) - \int_a^b f\, d\varphi\right| < \epsilon.$$

Consequently, we obtain

$$\left|S(f_\varphi,\wp') - \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} f\, d\varphi\right|$$

$$= \left|S(f_\varphi,\wp_0) - \sum_{j=1}^{k} S(f_\varphi,\wp_j) - \left(\int_a^b f\, d\varphi - \sum_{j=1}^{k}\int_{x_{j-1}}^{x_j} f\, d\varphi\right)\right|$$

$$\le \left|S(f_\varphi,\wp_0) - \int_a^b f\, d\varphi\right| + \sum_{j=1}^{k}\left|S(f_\varphi,\wp_j) - \int_{x_{j-1}}^{x_j} f\, d\varphi\right|$$

$$< \epsilon + \frac{k\,\epsilon}{k} = \epsilon + \epsilon = 2\epsilon.$$

Hence, the result is verified and we have

$$\left| S(f_\varphi, \wp') - \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f \, d\varphi \right| \leq 2\epsilon. \qquad \square$$

III. Conclusion

From the main results, we have some conclusions below.

1. The M_α $-$Stieltjes integral is well-defined as the generalization of the M_α integral. By taking $\varphi(x) = x$, the M_α integral is seen to be a special case of the M_α $-$Stieltjes integral.
2. The uniqueness of the M_α $-$Stieltjes integral's value and the linearity of both functions f and φ are proved.
3. The M_α $-$Stieltjes integral is valid for every subinterval and Saks-Henstock Lemma.

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