THE $M_\alpha$–STIELTJES INTEGRAL

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ABSTRACT. Based on the McShane integral, we define the $M_\alpha$–Stieltjes integral for real-valued function. In this paper, we will need the definition of an increasing function to bring the definition of the $M_\alpha$ integral into the $M_\alpha$–Stieltjes integral. The purpose of this paper is to define the $M_\alpha$–Stieltjes integral as a generalization of the $M_\alpha$ integral and investigate some of its basic properties such as uniqueness of the integral value, linearity, etc.

Keywords. McShane-Stieltjes integral, $M_\alpha$-partition, $M_\alpha$ integral, and increasing function

I. Introduction and Basic Definitions

There are two types of integral, those are descriptive integral and constructive integral. Both integrals develop rapidly. McShane integral is the one of constructive integral type constructed by McShane in 1960. Some authors like Jae Myung Park [4], Ju Han Yoon [6], and Russel A. Gordon [2] say that Lebesgue integral is equivalent to McShane integral. In 1997, Yoon [6] defined McShane-Stieltjes integral and proved some of its properties. Later on, Park [4] defined $M_\alpha$ integral based on McShane integral, and proved some properties. From those papers, the authors are motivated to discuss the $M_\alpha$–Stieltjes integral so that we can define the $M_\alpha$–Stieltjes integral as a generalization of the $M_\alpha$ integral and observe some properties.

Definition 1.1 (Park, 2010) Given $\varnothing = \{([x_{i-1}, x_i], \xi_i), 1 \leq i \leq n\}$, where $([x_{i-1}, x_i], \xi_i), 1 \leq i \leq n$ are interval-point pairs and $[x_{i-1}, x_i], 1 \leq i \leq n$ are non-overlapping sub-intervals of $[a, b]$ that is $[x_{i-1}, x_i]^0 \cap [x_{j-1}, x_j]^0 \neq \emptyset$, for $i \neq j$. Let $\delta$ be a positive function on $[a, b]$.

(1) $\varnothing$ is a $\delta$-fine McShane partition of $[a, b]$ if $[x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ and $\xi_i \in [a, b]$ for all $i = 1, 2, 3, \ldots, n$, and $\bigcup_{i=1}^n [x_{i-1}, x_i] = [a, b]$.

(2) $\varnothing$ is a $\delta$-fine $M_\alpha$ partition of $[a, b]$ for a constant $\alpha > 0$ if it is a $\delta$-fine McShane partition of $[a, b]$ and satisfying the $\sum_{i=1}^n \text{dist}([x_{i-1}, x_i], \xi_i) < \alpha$ where $\text{dist}([x_{i-1}, x_i], \xi_i) = \inf\{|t_i - \xi_i|: t_i \in [x_{i-1}, x_i]\}$. 

1
(3) A δ-fine Henstock partition of \([a,b]\) if \(\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))\) for all \(i = 1, 2, 3, \ldots, n\).

Let \(f: [a, b] \to R\). Given a δ-fine partition \(\mathcal{P} = \{([x_{i-1}, x_i], \xi_i), 1 \leq i \leq n\}\) we write 
\[S(f, \mathcal{P}) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i\]
for integral sum over \(\mathcal{P}\), where \(\Delta x_i = x_i - x_{i-1}\). Park [4] define the \(M_\alpha\) integral as follow,

**Definition 1.2** Let \(\alpha > 0\) be a constant. A function \(f: [a, b] \to R\) is \(M_\alpha\) integrable if there exists a real number \(L\) such that for each \(\epsilon > 0\) there is a positive function \(\delta: [a, b] \to R^+\) such that 
\[|S(f, \mathcal{P}) - L| < \epsilon \]
for each δ-fine \(M_\alpha\) partition \(\mathcal{P} = \{([x_{i-1}, x_i], \xi_i), 1 \leq i \leq n\}\) of \([a, b]\).

\(L\) is called the \(M_\alpha\) integral of \(f\) on \([a, b]\). Concern to the \(M_\alpha\)-Stieltjes integral, we need the concept of an increasing function \(\varphi\). A function \(\varphi: [a, b] \to R\) is called the increasing function on \([a, b]\) if \(x_1, x_2 \in [a, b]\) where \(x_1 < x_2\) then \(f(x_1) < f(x_2)\). By taking \(\varphi(x) = x\), the \(M_\alpha\) integral is seen to be a special case of the \(M_\alpha\)-Stieltjes integral.

**II. Main Results**

Based on the concept of \(M_\alpha\) integral and McShane-Stieltjes Integral, we define the \(M_\alpha\)-Stieltjes integral. Let \(P = \{([x_{i-1}, x_i], \xi_i), 1 \leq i \leq n\}\) is a δ-fine \(M_\alpha\) partition of \([a, b]\), then the \(M_\alpha\)-Stieltjes integral sum of \(f\) with respect to \(\varphi\) over \(\mathcal{P}\) is defined by 
\[S(f, \mathcal{P}) = \sum_{i=1}^{n} f(\xi_i)\Delta \varphi(x_i)\]
and the \(M_\alpha\)-Stieltjes integral of \(f\) with respect to \(\varphi\) on \([a, b]\) denoted by \(L = (M_\alpha S) \int_a^b f \ d\varphi\).

**Definition 2.1.** Let \(\varphi\) be an increasing function on \([a, b]\) and \(\alpha > 0\) be a constant. A function \(f: [a, b] \to R\) is \(M_\alpha\)-Stieltjes integrable with respect to \(\varphi\) if there exists a real number \(L\) with the following property: for each \(\epsilon > 0\) there is a positive function \(\delta: [a, b] \to R^+\) such that
\[|S(f, \mathcal{P}) - L| < \epsilon\]
for each δ-fine \(M_\alpha\) partition \(\mathcal{P} = \{([x_{i-1}, x_i], \xi_i), 1 \leq i \leq n\}\) of \([a, b]\).

\(L\) is called the \(M_\alpha\)-Stieltjes integral of \(f\) with respect to \(\varphi\) on \([a, b]\). The collection of all \(M_\alpha\)-Stieltjes integrable functions with respect to \(\varphi\) on \([a, b]\) denoted
by $M_\alpha S[a, b]$. We next verify the basic properties of the $M_\alpha$–Stieltjes integral which contained in these following Theorems.

**Theorem 2.1** Let $f: [a, b] \to R$ and $\varphi$ be an increasing function on $[a, b]$. If $f \in M_\alpha S[a, b]$ then it’s integral is unique.

*Proof.* Let $\epsilon > 0$. Since $f \in M_\alpha S[a, b]$, suppose that $K, L$ are the $M_\alpha$–Stieltjes integrals of $f$ written as $(M_\alpha S) \int_a^b f \, d\varphi = K$ and $(M_\alpha S) \int_a^b \lambda f \, d\varphi = L$ where $K \neq L$. There is a positive function $\delta_1: [a, b] \to R^+$ such that for each $\delta_1$–fine $M_\alpha$ partition $\varphi_1 = \{(x_{i-1}, x_i], \xi_i), 1 \leq i \leq n\} \subseteq [a, b]$, we have $|S(f, \varphi_1) - K| < \epsilon/2$, and there is a positive function $\delta_2: [a, b] \to R^+$ such that for each $\delta_2$–fine $M_\alpha$ partition $\varphi_2 = \{(x_{i-1}, x_i], \xi_i), 1 \leq i \leq n\} \subseteq [a, b]$, we have $|S(f, \varphi_2) - L| < \epsilon/2$. We define a positive function $\delta: [a, b] \to R^+$ where $\delta(\xi) = \min\{\delta_1(\xi), \delta_2(\xi)\}$ such that for each $\delta$–fine $M_\alpha$ partition satisfies:

$$|K - L| \leq |K - S(f, \varphi)| + |S(f, \varphi) - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$ 

Since $\epsilon > 0$, Theorem 2.2.9 on Bartle [1] proved that $K - L = 0$, it implies that $K = L$ which contradict to the assumption. Hence, the theorem is proved.

**Theorem 2.2 (Linear Space)** If $f, g \in M_\alpha S[a, b]$ and $\lambda \in R$, then $(f + g) \in M_\alpha S[a, b]$ and $\lambda f \in M_\alpha S[a, b]$, and satisfies:

1. $(M_\alpha S) \int_a^b (f + g) \, d\varphi = (M_\alpha S) \int_a^b f \, d\varphi + (M_\alpha S) \int_a^b \lambda \varphi \, d\varphi$
2. $(M_\alpha S) \int_a^b \lambda f \, d\varphi = \lambda (M_\alpha S) \int_a^b f \, d\varphi$

*Proof.* Let $\epsilon > 0$. (1) By hypothesis $f, g \in M_\alpha S[a, b]$, called $(M_\alpha S) \int_a^b f \, d\varphi = K$ and $(M_\alpha S) \int_a^b g \, d\varphi = L$. There is a positive function $\delta_1: [a, b] \to R^+$ such that for each $\delta_1$–fine $M_\alpha$ partition $\varphi_1 = \{(x_{i-1}, x_i], \xi_i), 1 \leq i \leq n\} \subseteq [a, b]$, we have $|S(f, \varphi_1) - K| < \epsilon/2$ and there is a positive function $\delta_2: [a, b] \to R^+$ such that for each $\delta_2$–fine $M_\alpha$ partition $\varphi_2 = \{(x_{i-1}, x_i], \xi_i), 1 \leq i \leq n\} \subseteq [a, b]$, we have $|S(g, \varphi_2) - L| < \epsilon/2$. Then we define a positive function $\delta: [a, b] \to R^+$ where $\delta(\xi) = \min\{\delta_1(\xi), \delta_2(\xi)\}$ such that for each $\delta$–fine $M_\alpha$ partition, we have:

$$|S((f + g)\varphi, \varphi) - (K + L)| = |\sum_{i=1}^n (f + g)(\xi_i)\Delta\varphi(x_i) - (K + L)|$$
Thus, of partition of partitions of for any such that

Proof. Let \( \epsilon > 0 \). Suppose \( f \in M_\alpha S[a, b] \), there is a positive function \( \delta: [a, b] \rightarrow R^+ \) such that for each \( \delta - \text{fine} M_\alpha \) partition \( \varphi = \{[x_{i-1}, x_i], \xi_i), 1 \leq i \leq n \} \) of \([a, b]\) and \( |S(f_\varphi, \varphi) - K| < \epsilon/|\lambda| \), we have

\[
|S(\lambda f_\varphi, \varphi) - \lambda K| = \sum_{i=1}^{n} \lambda f(\xi_i) \Delta \varphi(x_i) - \lambda K = |\lambda| \sum_{i=1}^{n} f(\xi_i) \Delta \varphi(x_i) - K < |\lambda| \frac{\epsilon}{|\lambda|} = \epsilon
\]

It implies that \( \lambda f \in M_\alpha S[a, b] \) and satisfies:

\[
(M_\alpha S) \int_{a}^{b} \lambda f \ d\varphi = \lambda K = \lambda (M_\alpha S) \int_{a}^{b} f \ d\varphi.
\]

From (1) and (2), these complete the proof of the theorem. \( \square \)

**Theorem 2.3.** Let \( f: [a, b] \rightarrow R \). If \( f \in M_\alpha S[a, b] \) and \([c, d] \subseteq [a, b]\) then \( f \in M_\alpha S[c, d] \).

**Proof.** Let \( \epsilon > 0 \). Suppose \( f \in M_\alpha S[a, b] \), there is a positive function \( \delta: [a, b] \rightarrow R^+ \) such that

\[
|S(f_\varphi, \varphi) - S(f_\varphi, Q)| < \epsilon
\]

for any \( \delta - \text{fine} M_\alpha \) partitions \( \varphi \) and \( Q \) of \([a, b]\). We define \( \varphi_1, \varphi_2 \) are \( \delta - \text{fine} M_\alpha \) partitions of \([c, d]\). \( \varphi_3 \) is \( \delta - \text{fine} M_\alpha \) partition of \([a, c]\), and \( \varphi_4 \) is \( \delta - \text{fine} M_\alpha \) partition of \([d, b]\). Then \( \varphi_3 \cup \varphi_1 \cup \varphi_2 \) and \( \varphi_3 \cup \varphi_2 \cup \varphi_4 \) are \( \delta - \text{fine} M_\alpha \) partitions of \([a, b]\). By using Theorem 2.4 we have:

\[
|S(f_\varphi, \varphi_1) - S(f_\varphi, \varphi_2)| = |S(f_\varphi, \varphi_3 \cup \varphi_1 \cup \varphi_4) - S(f_\varphi, \varphi_3 \cup \varphi_2 \cup \varphi_4)| < \epsilon.
\]

Thus, \( f \in M_\alpha S[c, d] \) and the theorem is proved. \( \square \)
Theorem 2.4. Let \( \varphi \) be an increasing function on \([a, b]\). Let \( f: [a, b] \to R \) and let \( c \in [a, b] \). If \( f \) is \( M_\alpha \)-Stieltjes integrable with respect to \( \varphi \) on each of the interval \([a, c]\) and \([c, b]\), then \( f \) is \( M_\alpha \)-Stieltjes integrable with respect to \( \varphi \) on \([a, b]\) and

\[
(M_\alpha S) \int_a^b f \, d\varphi = (M_\alpha S) \int_a^c f \, d\varphi + (M_\alpha S) \int_c^b f \, d\varphi.
\]

Proof. Let \( \epsilon > 0 \). By hypothesis \( f \in M_\alpha S[a, b] \). Since \( c \in [a, b] \), suppose that \( f \) is \( M_\alpha \)-Stieltjes integrable with respect to \( \varphi \) on each of the interval \([a, c]\) and \([c, b]\), called \((M_\alpha S) \int_a^c f \, d\varphi = K \) and \((M_\alpha S) \int_c^b f \, d\varphi = L \). There is a positive function \( \delta_1: [a, c] \to R^+ \) such that for each \( \delta_1 \)-fine \( M_\alpha \) partition \( \varphi_1 = \{([x_{i-1}, x_i], \xi_i), 1 \leq i \leq n\} \) of \([a, c]\), we have \( |S(f_\varphi, \varphi_1) - K| < \epsilon/2 \) and there is a positive function \( \delta_2: [c, b] \to R^+ \) such that for each \( \delta_2 \)-fine \( M_\alpha \) partition \( \varphi_2 = \{([x_{i-1}, x_i], \xi_i), 1 \leq i \leq n\} \) of \([c, b]\), we have \( |S(f_\varphi, \varphi_2) - L| < \epsilon/2 \). Then we define a positive function \( \delta: [a, b] \to R^+ \) where

\[
\delta(\xi) = \begin{cases} 
\min\{\delta_1(x), c - x\}, & \text{if } a \leq x < c \\
\min\{\delta_1(c), \delta_2(c)\}, & \text{if } x = c \\
\min\{\delta_2(x), x - c\}, & \text{if } c < x \leq b 
\end{cases}
\]

Let \( \varphi \) be a \( \delta \)-fine \( M_\alpha \) partition of \([a, b]\), then \( \varphi = \varphi_1 + \varphi_2 \). Since, \( S(f_\varphi, \varphi) = S(f_\varphi, \varphi_1) + S(f_\varphi, \varphi_2) \), we obtain

\[
|S(f_\varphi, \varphi) - (K + L)| = |S(f_\varphi, \varphi_1) + S(f_\varphi, \varphi_2) - (K + L)| \\
\leq |S(f_\varphi, \varphi_1) - K| + |S(f_\varphi, \varphi_2) - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Hence, the function \( f \) is \( M_\alpha \)-Stieltjes integrable with respect to \( \varphi \) on \([a, b]\) and

\[
(M_\alpha S) \int_a^b f \, d\varphi = K + L = (M_\alpha S) \int_a^c f \, d\varphi + (M_\alpha S) \int_c^b f \, d\varphi. \tag*{\square}
\]

Theorem 2.5. Let \( \varphi_1, \varphi_2 \) be some increasing functions on \([a, b]\).

1) If \( f \) is \( M_\alpha \)-Stieltjes integrable with respect to \( \varphi_1 \) and \( \varphi_2 \) on \([a, b]\), then \( f \) is \( M_\alpha \)-Stieltjes integrable with respect to \( (\varphi_1 + \varphi_2) \) and satisfies:

\[
(M_\alpha S) \int_a^b f \, d(\varphi_1 + \varphi_2) = (M_\alpha S) \int_a^b f \, d\varphi_1 + (M_\alpha S) \int_a^b f \, d\varphi_2.
\]

2) If \( f \) is \( M_\alpha \)-Stieltjes integrable with respect to \( \varphi \) and \( \lambda \in R^+ \) then \( f \) is \( M_\alpha \)-Stieltjes integrable with respect to \( \lambda \varphi \) and satisfies

\[
(M_\alpha S) \int_a^b f \, d(\lambda \varphi) = \lambda (M_\alpha S) \int_a^b f \, d\varphi.
\]
Proof. Let $\epsilon > 0$. 1) By hypothesis $f \in M_\alpha S[a, b]$, called $(M_\alpha S) \int_a^b f \, d\varphi_1 = K$ and $(M_\alpha S) \int_a^b f \, d\varphi_2 = L$. There is a positive function $\delta_1 : [a, b] \to R^+$ such that for each $\delta_1 - fine \ M_\alpha$ partition $\varphi_1 = \{(x_{i-1}, x_i], \xi_i), 1 \leq i \leq n\}$ of $[a, b]$, we have $|S(f_{\varphi_1}, \varphi_1) - K| < \epsilon/2$ and there is a positive function $\delta_2 : [a, b] \to R^+$ such that for each $\delta_2 - fine \ M_\alpha$ partition $\varphi_2 = \{(x_{i-1}, x_i], \xi_i), 1 \leq i \leq n\}$ of $[a, b]$, we have $|S(f_{\varphi_2}, \varphi_2) - L| < \epsilon/2$. Then we define a positive function $\delta : [a, b] \to R^+$ where $\delta(\xi) = \min\{\delta_1(\xi), \delta_2(\xi)\}$ such that for each $\delta - fine \ M_\alpha$ partition $\varphi$ of $[a, b]$, we have:

$$|S(f_{\varphi_1 + \varphi_2}, \varphi) - (K + L)| = |\sum_{i=1}^n f(\xi_i)\Delta(\varphi_1 + \varphi_2)(x_i) - (K + L)|$$

$$= \sum_{i=1}^n f(\xi_i)((\varphi_1 + \varphi_2)(x_i) - (\varphi_1 + \varphi_2)(x_{i-1})) - (K + L)$$

$$= \sum_{i=1}^n f(\xi_i)(\varphi_1(x_i) - \varphi_1(x_{i-1}) + \varphi_2(x_i) - \varphi_2(x_{i-1})) - (K + L)$$

$$= \sum_{i=1}^n f(\xi_i)\Delta\varphi_1(x_i) + \Delta\varphi_2(x_i)) - (K + L)$$

$$\leq \sum_{i=1}^n f(\xi_i)\Delta\varphi_1(x_i) - K + \sum_{i=1}^n f(\xi_i)\Delta\varphi_2(x_i) - L$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

It implies that $f_{\varphi_1 + \varphi_2} \in M_\alpha S[a, b]$ and satisfies:

$$(M_\alpha S) \int_a^b f \, d(\varphi_1 + \varphi_2) = K + L = (M_\alpha S) \int_a^b f \, d\varphi_1 + (M_\alpha S) \int_a^b f \, d\varphi_2.$$

2) By hypothesis $f \in M_\alpha S[a, b]$, $(M_\alpha S) \int_a^b f \, d\varphi = K$ and $\lambda$ is a positive constant.

There is a positive function $\delta : [a, b] \to R^+$ such that for each $\delta - fine \ M_\alpha$ partition $\varphi = \{(x_{i-1}, x_i], \xi_i), 1 \leq i \leq n\}$ of $[a, b]$ and $|S(f_{\lambda \varphi}, \varphi) - K| < \epsilon/\lambda$, we have

$$|S(f_{\lambda \varphi}, \varphi) - \lambda K| = \sum_{i=1}^n f(\xi_i)\lambda\Delta\varphi(x_i) - \lambda K = \lambda \sum_{i=1}^n f(\xi_i)\Delta\varphi(x_i) - K$$

$$< \lambda\frac{\epsilon}{\lambda} = \epsilon.$$

It implies that $f_{\lambda \varphi} \in M_\alpha S[a, b]$ and satisfies:

6
\[(M_\alpha S) \int_a^b f \, d\lambda \phi = \lambda K = \lambda (M_\alpha S) \int_a^b f \, d\phi.\]

From (1) and (2), the theorem is completed. \(\square\)

**Lemma 2.1. (Saks-Henstock Lemma)** Let \(f : [a, b] \rightarrow R\) be \(M_\alpha\)-Stieltjes integrable with respect to \(\phi\) on \([a, b]\). Let \(\epsilon > 0\), there exist a positive function \(\delta : [a, b] \rightarrow R^+\) such that for each \(\delta\)-fine \(M_\alpha\) partition \(\mathcal{G} = \{([x_{i-1}, x_i], \xi_i), 1 \leq i \leq n\}\) of \([a, b]\), we have

\[|S(f_\phi, \mathcal{G}) - \int_a^b f \, d\phi| < \epsilon.\]

Particularly, if \(\mathcal{G}' = \{([x_{i-1}, x_i], \xi_i), 1 \leq i \leq m\}\) is an arbitrary \(\delta\)-fine partial \(M_\alpha\) partition of \([a, b]\), we have

\[|S(f_\phi, \mathcal{G}') - \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f \, d\phi| \leq 2\epsilon.\]

**Proof.** We assume \(\mathcal{G}' = \{([x_{i-1}, x_i], \xi_i), 1 \leq i \leq m\}\) is an arbitrary \(\delta\)-fine partial \(M_\alpha\) partition of \([a, b]\), then the complement \([a, b] \setminus \bigcup_{i=1}^m [x_{i-1}, x_i]\) can be expressed as a fine collection of closed subintervals and we denote \([a, b] \setminus \bigcup_{i=1}^m [x_{i-1}, x_i] = \bigcup_{j=1}^k [x_{j-1}, x_j]\).

Since \(\epsilon > 0\) be arbitrary, there is a positive function \(\delta_j\) on \([x_{j-1}, x_j]\) such that if \(\mathcal{G}_j\) is a \(\delta_j\)-fine \(M_\alpha\) partition of \([x_{j-1}, x_j]\), we have

\[|S(f_\phi, \mathcal{G}_j) - \int_{x_{j-1}}^{x_j} f \, d\phi| < \epsilon/k.\]

Assume that \(\delta_j(\xi) \leq \delta(\xi)\) for all \(\xi \in [a, b]\). Let \(\mathcal{G}_0 = \mathcal{G}' \cup \mathcal{G}_1 \cup \mathcal{G}_2 \cup \ldots \cup \mathcal{G}_k\), obviously, \(\mathcal{G}_0\) is \(\delta\)-fine \(M_\alpha\) partition of \([a, b]\), we have

\[|S(f_\phi, \mathcal{G}_0) - \int_a^b f \, d\phi| = |S(f_\phi, \mathcal{G}') + \sum_{j=1}^k S(f_\phi, \mathcal{G}_j) - \int_a^b f \, d\phi| < \epsilon.\]

Consequently, we obtain

\[\left|S(f_\phi, \mathcal{G}') - \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f \, d\phi\right| = \left|S(f_\phi, \mathcal{G}_0) - \sum_{j=1}^k S(f_\phi, \mathcal{G}_j) - \left(\int_a^b f \, d\phi - \sum_{j=1}^k \int_{x_{j-1}}^{x_j} f \, d\phi\right)\right|\]

\[\leq \left|S(f_\phi, \mathcal{G}_0) - \int_a^b f \, d\phi\right| + \sum_{j=1}^k \left|S(f_\phi, \mathcal{G}_j) - \int_{x_{j-1}}^{x_j} f \, d\phi\right|.\]
\[ < \epsilon + \frac{k \epsilon}{k} = \epsilon + \epsilon = 2\epsilon. \]

Hence, the result is verified and we have

\[ \left| S(f_{\varphi}, \varphi') - \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f \, d\varphi \right| \leq 2\epsilon. \]

\[ \square \]

III. Conclusion

From the main results, we have some conclusions below.

1. The \( M_\alpha \)-Stieltjes integral is well-defined as the generalization of the \( M_\alpha \) integral.
   By taking \( \varphi(x) = x \), the \( M_\alpha \) integral is seen to be a special case of the \( M_\alpha \)-Stieltjes integral.

2. The uniqueness of the \( M_\alpha \)-Stieltjes integral’s value and the linearity of both functions \( f \) and \( \varphi \) are proved.

3. The \( M_\alpha \)-Stieltjes integral is valid for every subinterval and Saks-Henstock Lemma.

References


