On $P_2 \diamond P_n$-Supermagic Labeling of Edge Corona Product of Cycle and Path Graph

Riza Yulianto and Titin Sri Martini

Mathematics Department of Mathematics and Natural Sciences Faculty, Universitas Sebelas Maret, Surakarta, Indonesia

E-mail: yuliantoriza48@gmail.com, titinsmartini@gmail.com

Abstract. A simple graph $G = (V, E)$ admits a H-covering, where H is subgraph of G, if every edge in E belongs to a subgraph of G isomorphic to H. Graph G is H-magic if there is a total labeling $f : V(G) \bigcup E(G) \to 1, 2, ..., |V(G)| + |E(G)|$, such that each subgraph $H' = (V', E')$ of G isomorphic to H and satisfying $f(H') \overset{def}{=} \Sigma_{v \epsilon V'} f(v) + \Sigma_{e \epsilon E'} f(e) = m(f)$ where $m(f)$ is a constant magic sum. Additionaly, G admits H-supermagic if $f(V) = 1, 2, ..., |V|$. The edge corona $C_n \diamond P_n$ of C_n and P_n is defined as the graph obtained by taking one copy of C_n and n copies of P_n, and then joining two end-vertices of the i-th edge of C_n to every vertex in the i-th copy of P_n. This research aim is to find H-supermagic covering on an edge corona product of cycle and path graph $C_n \diamond P_n$ where H is $P_2 \diamond P_n$. We use k-balanced multiset to solve our reserarch. Here, we find that an edge corona product of cycle and path graph $C_n \diamond P_n$ is $P_2 \diamond P_n$ supermagic for $n \geq 3$.

1. Introduction

Let G be a simple graph $G = (V, E)$, where V is a set of vertices, and E is a set of edges. Chartrand and Lesniak [1] defined that cycle graph is a circuit with no repeated vertices, except the first and last vertices. The cycle graph with n vertices is denoted by C_n. They also defined path graph is a walk with no repeated vertices, path graph with n vertices is denoted by P_n.

Let G_1 and G_2 are two graphs on disjoint sets of n_1 and n_2 vertices, m_1 and m_2 edges, respectively. The edge corona $G_1 \diamond G_2$ is defined as the graph obtained by taking one copy of G_1 and m_1 copies of G_2, and then joining two end-vertices of the i-th edge of G_1 to every vertex in the i-th copy of G_2. Note that the edge corona $G_1 \diamond G_2$ of G_1 and G_2 has $n_1 + m_1 n_2$ vertices and $m_1 + 2m_1 n_2 + m_1 m_2$ edges, for detail defnition of graph see [4].

Gallian [2] defined a graph labeling as an assignment of integers to the vertices or edges, or both, subject to certain condition. Magic labelings was first introduced in 1963 by Sedláčk [9]. The concept of H-magic graphs was introduced in [3]. An edge-covering of a graph G is a family of different subgraphs $H_1, H_2, ..., H_k$ such that each edge of E belongs to at least one of the subgraphs H_i, $1 \leq i \leq k$. Then, it is said that G admits an $(H_1, H_2, ..., H_k)$-edge covering. If every H_i is isomorphic to a given graph H, then we say that G admits an H-covering. Suppose that $G = (V(G), E(G))$ admits an H-covering. A bijective function $f : V(G) \cup E(G) \to \{1, 2, ..., |V(G)| + |E(G)|\}$ is an H-magic labeling of G if there exist a positive integer $m(f)$, which we call magic sum such that for each subgraph $H' = (V(G)', E(G)')$ of G isomorphic H, $f(H') = \sum_{v \in V(G)'} f(v) + \sum_{e \in E(G)'} f(e) = m(f)$. In this case we say that the graph G is H-magic. When $f(v) = \{1, 2, ..., |V(G)|\}$, then G is H-supermagic and we denote supermagic-sum is $s(f)$.

In [3], they proved that a complete bipartite graph $K_{n,n}$ could be covered by magic star covering $K_{1,n}$. Then Lladó and Moragas [5] proved in [3] the same graph containing a cycle cover, they also proved that C_3-supermagic labelings on a wheel graph W_n for $n \geq 5$ odd and C_4-supermagic labeling of a prism graph and a book graph. Marbun and Salman [6] then proved that W_n-supermagic labelings for a wheel W_n k-multilevel corona with a cycle C_n.

In this paper, we study an H-supermagic labeling of edge corona product of cycle and path graph. We prove that a edge corona product of cycle C_n and path P_n graph has a $P_2 \diamond P_n$-supermagic labeling for $n \geq 3$.

2. Main Result

A multiset is a set that allows the existence of same elements in it(Maryati et al. [7]). Let X be a set containing some positive integers. We use the notation $[a, b]$ to mean $\{x \in \mathbb{N} | a \leq x \leq b\}$ and ΣX to mean $\sum_{x \in X} x$. For any $k \in \mathbb{N}$, the notation $k + [a, b]$ means $k + x | x \in [a, b]$. According to Guitérrez and Llado [3], the set X is k-equipartion if there exist k subsets of X. say $X_1, X_2, \ldots, X_k$ such that $\bigcup_{i=1}^{k} X_i = X$ and $|X_i| = \frac{|X|}{k}$ for every $i \in [1, k]$.

*2.1. k-**balanced multiset***

In this research, we used technique k-balancemultiset that introduced by Maryati et al. [7]. Let Y be a multiset of positive integers and $k \in \mathbb{N}$. A multiset Y is k-balanced if there are k subsets of Y where $Y_i = Y_1 = Y_2 = ... = Y_k$ then for each $i \in [1, k]$. We obtain $|Y_i| = |Y_k|, \sum Y_i = \frac{\sum Y}{k} \in N$ and $\biguplus_{i=1}^{k} Y_i = Y$.

Lemma 2.1 *[8] Let x, y, and k be integers, such that $1 \leq x \leq y$ and $k > 1$. If $X = [x, y]$ and $|X|$ is a multiple $2k$, then X is k-balanced.*

Here, we have several lemmas on k-balanced multiset to build theorem.

Lemma 2.2 *Let k and x be positive integers $k \geq 3$. Let $Y = [1, k] \uplus [1, k] \uplus [x + 1, x + k]$, then Y is k-balanced.*

Proof. For every $i \in [1, k]$ we define the multisets $Y_i = \{a_i, b_i, c_i\}$ with

$$\begin{aligned} a_i &= \left\lfloor \frac{i+1}{2} \right\rfloor && \text{for} \quad i \in [1, k] \\ b_i &= \left\lceil \frac{i+k}{2} \right\rceil && \text{for} \quad i \in [1, k] \\ c_i &= x + k + 1 - i && \text{for} \quad i \in [1, k]. \end{aligned}$$

Then, defined set

$$\begin{aligned} A &= \{a_i | 1 \leq i \leq k\} = [1, k] \\ B &= \{b_i | 1 \leq i \leq k\} = [1, k] \\ C &= \{c_i | 1 \leq i \leq k\} = [x + 1, x + k]. \end{aligned}$$

Since $A \uplus B \uplus C = Y$ and $\biguplus_{i=1}^{k} Y_i = Y$, $|Y_i| = 3$ and $\sum Y_i = x + \frac{3k+3}{2}$ for every $i \in [1, k]$, so we have Y is k-balanced.

Lemma 2.3 *Let k and x be positive integers $k \geq 3$. If $Z = [x + 1, x + k^2]$ and $|Z|$ is k^2, then Z is k-balanced.*

Proof. For every $i \in [1,k]$ we define the multisets $Z_i = \{a_j^i | 1 \leq j \leq k\}$ where

$$a_j^i = \begin{cases} x+i, & \text{for} \quad i \in [1,k] \textit{ and } j=1; \\ a_{j-1}^i + 1, & \text{for} \quad j+i = k+2; \\ a_{j-1}^i + x + 1, & \text{for} \quad i \textit{ and } j \textit{ others}. \end{cases}$$

Since $|Z_i| = k$; $\biguplus_{i=1}^k Z_i = Z$ and $\sum Z_i = (x+k^2)\frac{k+1}{2}$ for every $i \in [1,k]$ then Z is k-balanced.

Lemma 2.4 *Let x, y and k be positive integers $k \geq 4$. If $W = [1,x] \uplus [1,x] \uplus [x+1, x+k] \uplus [y+1, y+k]$, then W is k-balanced.*

Proof. For every $i \in [1,k]$ we define the multisets $W_i = \{a_i, b_i, c_i, d_i\}$ with

$$\begin{aligned} a_i &= i & \text{for} \quad & i \in [1,k] \\ b_i &= \begin{cases} 1+i, & \text{for} \quad i \in [1,k-1]; \\ 1, & \text{for} \quad i = k; \end{cases} \\ c_i &= \begin{cases} x+k-i, & \text{for} \quad i \in [1,k-1]; \\ x+k, & \text{for} \quad i = k; \end{cases} \\ d_i &= y+k+1-i & \text{for} \quad & i \in [1,k] \end{aligned}$$

Then, defined set

$$\begin{aligned} A &= \{a_i | 1 \leq i \leq k\} = [1,k] \\ B &= \{b_i | 1 \leq i \leq k\} = [1,k] \\ C &= \{c_i | 1 \leq i \leq k\} = [x+1, x+k] \\ D &= \{d_i | 1 \leq i \leq k\} = [y+1, y+k]. \end{aligned}$$

Since $A \uplus B \uplus C \uplus D = W$ and $\biguplus_{i=1}^k W_i = W$, $|W_i| = 4$ and $\sum W_i = 5k+2$ for every $i \in [1,k]$ then W is k-balanced.

2.2. $P_2 \diamond P_n$-Supermagic Labeling on A Cycle Graph Edge Corona with Path $C_n \diamond P_n$

The edge corona product between C_n and P_n, denoted by $C_n \diamond P_n$ is a graph obtained by taking one copy of C_n and $|E(C_n)|$ copies of P_n and then joining two end-vertices of the i-th edge of C_n to every vertex in the i-th copy of P_n.

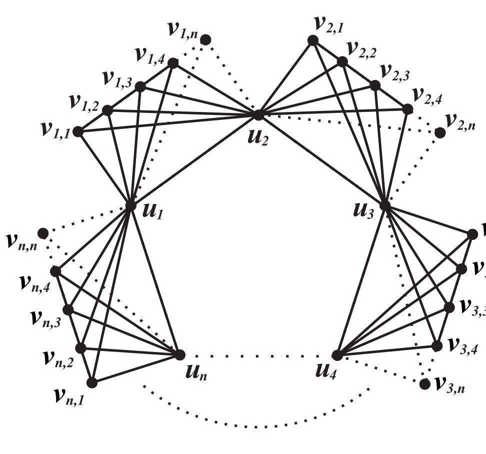

Figure 1. A Cycle Graph Edge Corona with Path $C_n \diamond P_n$

Theorem 2.1 *Let n be positive integers with $n \geq 3$. A graph $C_n \diamond P_n$ is $P_2 \diamond P_n$-supermagic.*

Proof. Let G be a $C_n \diamond P_n$ graph for any integer $n \geq 3$. Then $|V(G)| = n(n+1)$ and $|E(G)| = 3n^2$. Let $A = [1, 4n^2 + n]$. We define a bijective function $f : V(G) \cup E(G) \to \{1, 2, ..., 4n^2 + n\}$.

Here we have two cases to be considered.

Case 1. For n odd. Let $V(G) = \{v_i; 0 \leq i \leq n\} \uplus \{u^i_j; 0 \leq i \leq n, 0 \leq j \leq n\}$ and $E(G) = \{v_0v_1, v_1v_2, \ldots v_nv_0\} \uplus \{e^i_j; 0 \leq i \leq n, 0 \leq j \leq n\}$. Given a set of labels for all vertices and edges of G denoted by A where $A = [1, 4n^2 + n]$. Partition A into 3 sets, $A = X \uplus Y \uplus Z$, where $X = [1, n] \uplus [1, n] \uplus [n^2 + n + 1, n^2 + 2n]$, $Y = [n + 1, n^2 + n]$, and $Z = [n^2 + 2n + 1, 4(n^2) + n]$. Then we define the total labeling f on G as follows. First, partition X into 2 sets: $X_1 = [1, n]$ and $X_2 = [n^2 + n + 1, n^2 + 2n]$. The vertices v_i where $0 \leq i \leq n$ are labeled by set X_1 and edges $\{v_0v_1, v_1v_2, \ldots v_nv_0\}$ are labeled by set X_2. According to Lemma 2.2, if $x = n^2 + n$ and $k = n$ we have n-balanced. Let $X_1 \uplus X_2$, then $\uplus^n_{i=1} X_i = X$ and we have $\sum X_i = \frac{2n^2+5n+3}{2}$. The vertices u^i_j where $0 \leq i \leq n$ and $0 \leq j \leq n$ are labeled by set Y. Define that u^i_j are vertices on path. According Lemma 2.3, if $x = n$, $k = n$, and $|Y| = n^2$ we have n-balanced where $\sum Y_i = \frac{n^3+2n^2+n}{2}$. Then, the edges e^i_j are labeled by set Z where e^i_j are edges on path and edge on product edge coronation. According Lemma 2.1, if $x = n^2 + 2n + 1$, $y = 4n^2 + n$, and $|Z| = 3n^2 - n$ we have n-balanced where $\sum Y_i = \frac{15n^3+4n^2-1}{2}$.

Case 2. For n even. Let $V(G) = \{v_i; 0 \leq i \leq n\} \uplus \{u_i; 0 \leq i \leq n\}$ and $E(G) = \{e^i_j; 0 \leq i \leq n, 0 \leq j \leq n\}$. Partition A into 3 sets, $A = P \uplus Q \uplus R$, where $P = [1, n] \uplus [1, n] \uplus [n+1, 2n] \uplus [2n+1, 3n]$, $Q = [3n+1, n^2+n)]$, dan $R = [n^2+n+1, 4n^2+n]$. Then we define the total labeling f on G as follows. First, partition P into 3 sets: $P_1 = [1, n]$, $P_2 = [n + 1, 2n]$, and $P_3 = [2n + 1, 3n]$. The vertices v_i where $0 \leq i \leq n$ are labeled by set P_1, P_2, and P_3. According to Lemma 2.4, if $x = n$ and $k = n$ we have n-balanced. Let $P_1 \uplus P_2 \uplus P_3$, then $\biguplus^n_{i=1} P_i = P$ and we have $\sum P_i = 5n+2$. The vertices u_i where $0 \leq i \leq n$ are labeled by set Q. Define that u_i are two vertices in path. According Lemma 2.1, if $x = 3n+1$, $y = n^2+n$, and $|Q| = 2n$ we have n-balanced where $\sum Q_i = n^2+4n+1$. Then, the edges e^i_j are labeled by set R where e^i_j are edges on graph G. According Lemma 2.1, if $x = n^2 + n + 1$, $y = 4n^2 + n$, and $|R| = 3n^2$ we have n-balanced where $\sum R_i = \frac{15n^3+6n^2+3n}{2}$.

Furthermore, the constant supermagic sum of a subgraph $P_2 \diamond P_n$ are as follows

$$f(P_2 \diamond P_n) = \begin{cases} 8n^3 + 4n^2 + 3n + 1, & \text{for } n \text{ odd;} \\ \frac{15n^3+8n^2+21n+6}{2}, & \text{for n even.} \end{cases}$$

□

Figure 2 illustrates an example of $P_2 \diamond P_3$-supermagic labeling on $C_3 \diamond P_3$ graph.

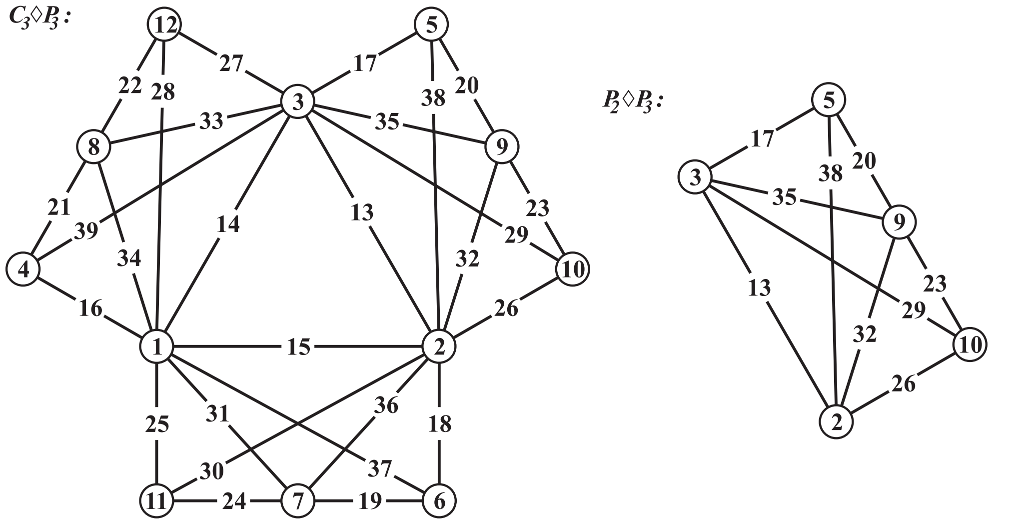

Figure 2. A $P_2 \diamond P_3$-supermagic labeling on $C_3 \diamond P_3$ graph

3. Conclusion

In this paper we have shown the $P_2 \diamond P_n$-supermagic labeling of edge corona product of cycle and path graph.

Open Problem: For further research we can studied $P_2 \diamond P_n$-supermagic labeling on $C_n \diamond P_m$ with $n \geq 3$ and $m \geq 2$.

Acknowledgments

We gratefully ackowledge the support from Mathematics Department of Mathematics and Natural Sciences Faculty, Universitas Sebelas Maret.

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