On the Total Edge Irregularity Strength of Generalized Butterfly Graph

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Abstract. Let $G(V, E)$ be a connected, simple, and undirected graph with vertex set $V$ and edge set $E$. A total $k$-labeling is a map that carries vertices and edges of a graph $G$ into a set of positive integer labels $\{1, 2, ..., k\}$. An edge irregular total $k$-labeling is a labeling such that the weights calculated for all edges are distinct. The weight of an edge $uv$ in $G$, denoted by $wt(uv)$, is defined as the sum of the label of $u$, the label of $v$, and the label of $uv$. The total edge irregularity strength of $G$, denoted by $tes(G)$, is the minimum value of the largest label $k$ over all such edge irregular total $k$-labelings. A generalized butterfly graph, $BF_n$, obtained by inserting vertices to every wing with assumption that sum of inserting vertices to every wing are same then it has $2n + 1$ vertices and $4n - 2$ edges. In this paper, we investigate the total edge irregularity strength of generalized butterfly graph, $BF_n$, for $n \geq 2$. The result is $tes(BF_n) = \lceil \frac{4n}{3} \rceil$.

1. Introduction

Let $G(V, E)$ be a connected, simple, and undirected graph with vertex set $V$ and edge set $E$. A labeling of a graph $G$ is a mapping that carries a set of graph elements into a set of positive integers, called labels (Wallis [8]). If the domain of mapping is a vertex set, or an edge set, or a union of vertex and edge sets, then the labeling is called vertex labeling, edge labeling, or total labeling, respectively. In Gallian’s survey [2], he showed that there were various kinds of labelings on graphs, and one of them was an irregular total labeling.

Baća et al. [1] introduced the notion of a total $k$-labeling, an edge irregular total $k$-labeling, and a vertex irregular total $k$-labeling. For a graph $G(V, E)$, they defined a labeling $f : V \cup E \to \{1, 2, ..., k\}$ to be a total $k$-labeling. An edge irregular total $k$-labeling with vertex set $V$ and edge set $E$ is a labeling $\lambda : V(G) \cup E(G) \to \{1, 2, ..., k\}$ such that for two different edges $e = u_iv_j$ and $f = u_kv_l$ then their weight $wt(e) \neq wt(f)$. The weight of an edge $uv$ in $G$, denoted by $wt(uv)$, is defined as the sum of the label of $u$, the label of $v$, and the label of $uv$, that is

$$wt(uv) = \lambda(u) + \lambda(uv) + \lambda(v).$$

Furthermore, Baća et al. [1] also defined the total edge irregularity strength of $G$, denoted by $tes(G)$, as the minimum value of the largest label $k$ over all such edge irregular total $k$-labelings. They also gave a lower bound and an upper bound on $tes(G)$ with vertex set $V$ and a non-empty edge set $E$,

$$\left\lfloor \frac{|E| + 2}{3} \right\rfloor \leq tes(G) \leq |E|. \tag{1}$$
Many researchers have investigated $tes(G)$ to some graph classes. By then, there are some result related to the $tes(G)$. Bača et al. [1] proved $tes(G)$ of paths, cycles, stars, wheels, and friendship graphs, that are, $tes(P_n) = tes(C_n) = \lceil \frac{n+2}{3} \rceil$, $tes(S_n) = \lceil \frac{n+1}{3} \rceil$, $tes(W_n) = \lceil \frac{2n+2}{3} \rceil$ for $n \geq 3$, and $tes(F_n) = \lceil \frac{3n+2}{3} \rceil$. Then, $tes(G)$ of tree $G$ with edge set $E$ and maximum degree $\Delta$ had been found by Ivančo and Jendrol’ [6], that is $tes(G) = \max\{\lceil \frac{\Delta+1}{3} \rceil, \lceil \frac{|E|+2}{3} \rceil\}$. In 2008, Nurdin et al. [7] proved $tes(G)$ of the corona product of paths with some graphs, namely paths $P_n$, cycles $C_n$, stars $S_n$, gears $G_n$, friendships $F_n$, and wheels $W_n$. Haque [3] in 2012 proved $tes(G)$ of generalized Petersen graphs $P(n,k)$ and Indriati et al. [4] found $tes(G)$ of Helm, $H_n$, and disjoint union of $t$ isomorphic helms, $tH_n$. In 2013 Indriati et al. [5] found $tes(H^1_n) = \lceil \frac{4n+2}{3} \rceil$, $tes(H^2_n) = \lceil \frac{5n+2}{3} \rceil$, and $tes(H^m_n) = \lceil \frac{(m+3)n+2}{4} \rceil$ for $n \geq 3$ and $m \equiv 0 \pmod{3}$.

Weisstein [9] defined the butterfly graph is a planar undirected graph with 5 vertices and 6 edges. In this paper, we investigate the total edge irregularity strength of generalized butterfly graph, $BF_n$, for $n \geq 2$.

2. Main Result

A generalized butterfly graph, $BF_n$, obtained by inserting vertices to every wing with assumption that sum of inserting vertices to every wing are same then it has $2n+1$ vertices and $4n-2$ edges. Let the vertex set of $BF_n$ be $V(BF_n) = \{v_i \mid i = 0, 1, 2, ..., 2n\}$ and the edge set of $BF_n$ be $E(BF_n) = \{(v_i, v_{i+1}) \mid i = 1, 2, ..., n-1, n+1, ..., 2n-1\} \cup \{(v_0, v_i) \mid i = 1, 2, ..., 2n\}$. Figure 1 illustrates the generalized butterfly graph $BF_n$. Based on Figure 1, $BF_n$ has $\{v_0\}$ as an apex, $\{v_1, v_2, v_3, ..., v_{(n-1)}, v_n\}$ as vertices on right wing, and $\{v_{(n+1)}, v_{(n+2)}, v_{(n+3)}, ..., v_{(2n-1)}, v_{2n}\}$ as vertices on left wing.

![Figure 1. The generalized butterfly graph $BF_n$](image)

In the next theorem, we present the total edge irregularity strength of generalized butterfly graph, $BF_n$, for $n \geq 2$ as follows:

**Theorem 2.1** For $n \geq 2$, $tes(BF_n) = \lceil \frac{4n}{3} \rceil$.

**Proof.** From the lower bound of total edge irregularity strength we have that $tes(BF_n) \geq \lceil \frac{4n}{3} \rceil$, $n \geq 2$. To prove the equality, it is sufficient to show the existence of an edge irregular total $k_1$-labeling with $k_1 = \lceil \frac{4n}{3} \rceil$. Let $k_1 = \lceil \frac{4n}{3} \rceil$. Then from inequality (1) it follows that, $tes(BF_n) \geq \lceil \frac{|E(BF_n)|+2}{3} \rceil = \lceil \frac{(4n-2)+2}{3} \rceil = \lceil \frac{4n}{3} \rceil = k_1$, that is $tes(BF_n) \geq k_1$. To prove the reverse
inequality, we define a function \( f_1 \) as follows.

**Case 1:** For \( n \equiv 2(\text{mod} 3), \ n \geq 2. \)

\[
f_1(v_0) = \left\lfloor \frac{4n}{3} \right\rfloor.
\]

\[
f_1(v_i) = \begin{cases} 1, & \text{for } 1 \leq i \leq \frac{2n+5}{3}; \\ f_1(v_{i-1}) + (-1)^i(\frac{2n-1}{3} - i(\text{mod } 2)), & \text{for } \frac{2n+5}{3} \leq i \leq 2n. \end{cases}
\]

\[
f_1(v_{vi}) = \begin{cases} i, & \text{for } 1 \leq i \leq \frac{2n+2}{3}; \\ 2, & \text{for } \frac{2n+5}{3} \leq i \leq n - 1; \\ 1, & \text{for } n + 1 \leq i \leq 2n - 1. \end{cases}
\]

\[
f_1(v_{0vi}) = \begin{cases} \frac{2n-4}{3} + i, & \text{for } 1 \leq i \leq \frac{2n+5}{3}; \\ \left\lceil \frac{4n}{3} \right\rceil, & \text{for } \frac{2n+8}{3} \leq i \leq 2n. \end{cases}
\]

**Case 2:** For \( n \equiv 0(\text{mod} 3), \ n \geq 3 \) and \( n \) is odd.

\[
f_1(v_0) = \left\lfloor \frac{4n}{3} \right\rfloor.
\]

\[
f_1(v_i) = \begin{cases} \frac{2n}{3}, & \text{for } 1 \leq i \leq \frac{2n}{3}; \\ f_1(v_{i-1}) + (-1)^{i-1}(\frac{2n-3}{3} - i(\text{mod } 2)), & \text{for } \frac{2n+6}{3} \leq i \leq n; \\ f_1(v_{i-1}) + 1, & \text{for } i = n + 1; \\ f_1(v_{i-1}) + (-1)^i(\frac{2n+3}{3} - i(\text{mod } 2)), & \text{for } n + 2 \leq i \leq 2n. \end{cases}
\]

\[
f_1(v_{vi}) = \begin{cases} 1, & \text{for } i = \frac{2n}{3} \text{ and } n + 1 \leq i \leq 2n - 1; \\ 2, & \text{for } \frac{2n+3}{3} \leq i \leq n - 1; \end{cases}
\]

\[
f_1(v_{0vi}) = \begin{cases} \frac{2n-3}{3} + i, & \text{for } 1 \leq i \leq \frac{2n+3}{3}; \\ \left\lceil \frac{4n}{3} \right\rceil, & \text{for } \frac{2n+6}{3} \leq i \leq 2n. \end{cases}
\]

**Case 3:** For \( n \equiv 1(\text{mod } 3), \ n \geq 4. \)

\[
f_1(v_0) = \left\lfloor \frac{4n}{3} \right\rfloor.
\]
It can be seen that the function $f_1$ is a map from $V(BF_n) \cup E(BF_n)$ into $\{1, 2, ..., \lceil \frac{4n}{3} \rceil \}$. Therefore, $f_1$ is a total $k_1$-labeling with $k_1 = \lceil \frac{4n}{3} \rceil$.

We observe that the weights of the edges are:

$$wt(v_iv_{i+1}) = \begin{cases} 2 + i, & \text{for } 1 \leq i \leq n - 1; \\ 1 + i, & \text{for } n + 1 \leq i \leq 2n - 1. \end{cases}$$

**Case 1:** For $n \equiv 2(mod\ 3)$, $n \geq 2$.

$$wt(v_0v_i) = \begin{cases} 2n + i, & \text{for } 1 \leq i \leq \frac{2n+5}{3}; \\ wt(v_0v_{i-1}) + (-1)^i(\frac{2n-1}{3} - i(mod\ 2)), & \text{for } \frac{2n+8}{3} \leq i \leq 2n. \end{cases}$$
Case 2: For \( n \equiv 0(\text{mod } 3) \), \( n \geq 3 \) and \( n \) is odd.

\[
\text{wt}(v_0v_i) = \begin{cases} 
2n + i, & \text{for } 1 \leq i \leq \frac{2n}{3}; \\
wt(v_0v_{i-1}) + (-1)^{i-1}(\frac{2n-3}{3} + i(\text{mod } 2)), & \text{for } \frac{2n+3}{3} \leq i \leq n; \\
wt(v_0v_{i-1}) + 1, & \text{for } i = n + 1; \\
wt(v_0v_{i-1}) + (-1)^i(\frac{2n+3}{3} - i(\text{mod } 2)), & \text{for } n + 2 \leq i \leq 2n.
\end{cases}
\]

Case 3: For \( n \equiv 1(\text{mod } 3) \), \( n \geq 4 \).

\[
\text{wt}(v_0v_i) = \begin{cases} 
2n + i, & \text{for } 1 \leq i \leq \frac{2n}{3}; \\
wt(v_0v_{i-1}) + (-1)^{i-1}(\frac{2n-2}{3} - i(\text{mod } 2)), & \text{for } 6 \leq i \leq 8 \text{ and } n = 4; \\
wt(v_0v_{i-1}) + 1, & \text{for } i = 8 \text{ and } n = 7; \\
wt(v_0v_{i-1}) + (-1)^{i-1}(\frac{2n-5}{3} + i(\text{mod } 2)), & \text{for } 9 \leq i \leq 14 \text{ and } n = 7; \\
wt(v_0v_{i-1}) + 1, & \text{for } \frac{2n+10}{3} \leq i \leq \frac{2n+13}{3} \text{ and } n \geq 10; \\
wt(v_0v_{i-1}) + (-1)^i(\frac{2n-5}{3} - i(\text{mod } 2)), & \text{for } \frac{2n+16}{3} \leq i \leq 2n \text{ and } n \geq 10.
\end{cases}
\]

It can be seen that the weights of edges of \( BF_n \) under the total \( k_1 \)-labeling, \( f_1 \), form consecutive integers from 3 to \( 4n \). It means that the weights of all edges are distinct, then we have \( 	ext{tes}(BF_n) \leq k_1 \). This completes the proof. So, the labeling is an edge irregular total \( k_1 \)-labeling with \( k_1 = \lceil \frac{4n}{3} \rceil \). Therefore, \( 	ext{tes}(BF_n) = \lceil \frac{4n}{3} \rceil \), for \( n \geq 2 \). \( \square \)

Figure 2 shows an edge irregular total labeling of \( BF_8 \).

![Figure 2. An edge irregular total 11-labeling of BF_8](image-url)
3. Concluding Remark

Furthermore, we conclude this paper with the following open problem for the direction of further research which is still in progress.

**Open Problem**: Determine the total edge irregularity strength of generalized butterfly graph $BF_n$ for $n \geq 3$ and $n$ is odd. Then, generalized butterfly graph isomorphic with broken fan graph. Broken fan graph is fan graph which divide into $n$ parts for $n = 2$. So, determine the total edge irregularity strength of broken fan graph which divide into $n$ parts for $n \geq 3$.

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