On the strong metric dimension of antiprism graph, king graph, and $K_m \odot K_n$ graph

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Abstract. Let G be a connected graph with a set of vertices $V(G)$ and a set of edges $E(G)$. The interval $I[u, v]$ between u and v to be the collection of all vertices that belong to some shortest u-v path. A vertex $s \in V(G)$ is said to be strongly resolved for vertices u, $v \in V(G)$ if $v \in I[u, s]$ or $u \in I[v, s]$. A vertex set $S \subseteq V(G)$ is a strong resolving set for G if every two distinct vertices of G are strongly resolved by some vertices of S. The strong metric dimension of G, denoted by $sdim(G)$, is defined as the smallest cardinality of a strong resolving set. In this paper, we determine the strong metric dimension of an antiprism A_n graph, a king $K_{m,n}$ graph, and a $K_m \odot K_n$ graph. We obtain the strong metric dimension of an antiprim graph A_n are n for n odd and $n + 1$ for n even. The strong metric dimension of King graph $K_{m,n}$ is $m + n - 1$. The strong metric dimension of $K_m \odot K_n$ graph are n for $m = 1$, $n \geq 1$ and $mn - 1$ for $m \geq 2$, $n \geq 1$.

1. Introduction

The strong metric dimension was introduced by Sebö and Tannier [6] in 2004. Let G be a connected graph with a set of vertices $V(G)$ and a set of edges $E(G)$. Oelermann and Peters-Fransen [5] defined the interval $I[u, v]$ between u and v to be the collection of all vertices that belong to some shortest $u - v$ path. A vertex $s \in S$ is said to strongly resolve two vertices u and v if $u \in I[v, s]$ or $v \in I[u, v]$. A vertex set S of G is a strong resolving set for G if every two distinct vertices of G are strongly resolved by some vertices of S. The strong metric basis of G is a strong resolving set with minimal cardinality. The strong metric dimension of a graph G is defined as the cardinality of strong metric basis denoted by $sdim(G)$.

Some authors have investigated the strong metric dimension to some graph classes. Sebö and Tannier [6] observed the strong metric dimension of complete graph K_n, cycle graph C_n, and tree. Kratica et al. [2] observed the strong metric dimension of *hamming* graph $H_{n,k}$. At the same year, Kratica et al [3] determined the strong metric dimension of convex polytope D_n and T_n. Yi [8] determined that $sdim(G) = 1$ if only if G is path graph and $sdim(G) = n-1$ if only if G is complete graph. Kusmayadi et al. [4] determined the strong metric dimension of sunflower graph, t-fold wheel graph, helm graph, and friendship graph. In this paper, we determine the strong metric dimension of an antiprism graph, a king graph, and a $K_m \odot K_n$ graph.

2. Main Results

2.1. Strong Metric Dimension

Let G be a connected graph with a set of vertices $V(G)$, a set of edges $E(G)$, and $S=\{s_1, s_2, \ldots, s_k\} \in V(G)$. Oelermann and Peters-Fransen [5] defined the interval $I[u, v]$ between u and v to be the collection of all vertices that belong to some shortest $u - v$ path. A vertex $s \in S$ is said to strongly resolve two vertices u and v if $u \in I[v, s]$ or $v \in I[u, s]$. A vertex set S of G is a strong resolving set for G if every two distinct vertices of G are strongly resolved by some vertices of S. The strong metric basis of G is a strong resolving set with minimal cardinality. The strong metric dimension of a graph G is defined as the cardinality of strong metric basis denoted by $sdim(G)$. We often make use of the following lemma and properties about strong metric dimension given by Kratica et al. [3].

Lemma 2.1 *Let $u,\ v \in\ V(G),\ u \neq\ v,$*

(i) $d(w,v) \leq\ d(u,v)$ for each w such that $u\ w \in\ E(G)$, and

(ii) $d(u,w) \leq\ d(u,v)$ for each w such that $v\ w \in\ E(G)$.

Then there does not exist vertex $a \in\ V(G),\ a \neq\ u,v$ that strongly resolves vertices u and v.

Property 2.1 *If $S \subset\ V(G)$ is strong resolving set of graph G, then for every two vertices $u,\ v \in\ V(G)$ satisfying conditions* 1 *and* 2 *of Lemma* 2.1*, obtained $u \in\ S$ or $v \in\ S$.*

Property 2.2 *If $S \subset\ V(G)$ is strong resolving set of graph G, then for every two vertices $u,\ v \in\ V(G)$ satisfying $d(u,\ v) =\ diam(G)$, obtained $u \in\ S$ or $v \in\ S$.*

2.2. The Strong Metric Dimension of antiprism graph

Bača [1] defined the antiprism graph A_n for $n \geq 3$ is a 4-regular graph with $2n$ vertices and $4n$ edges. It consists of outer and inner C_n, while the two cycles connected by edges v_iu_i and $v_iu_{1+i(mod\ n)}$ for i = 1, 2, 3, ..., n. The antiprism graph A_n can be depicted as in Figure 1.

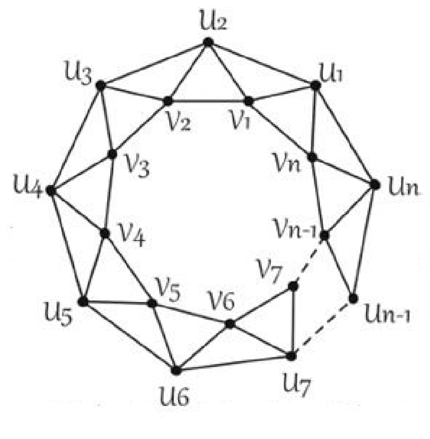

Figure 1. Antiprism graph A_n

Lemma 2.2 *For every integer $n \geq 3$ and n odd, if S is a strong resolving set of antiprism graph A_n then $\mid S \mid\ \geq n$.*

Proof. We know that S is a strong resolving set of antiprism graph A_n. Suppose that S contains at most n - 1 vertices, then $\mid S \mid < n$. Let V_1, $V_2 \subset V(A_n)$, with $V_1 = \{u_1, u_2, \ldots, u_n\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$. Now, we define $S_1 = V_1 \cap S$ and $S_2 = V_2 \cap S$. Without loss of generality, we may take $\mid S_1 \mid = p$, $p > 0$ and $\mid S_2 \mid = q$, $q \geq 0$. Clearly $p + q \geq n$, if not then there are two distinct vertices v_a and v_b where $v_a \in V_1 \setminus S_1$ and $v_b \in V_2 \setminus S_2$ such that for every $s \in S$, we obtain $v_a \notin I[v_b, s]$ and $v_b \notin I[v_a, s]$. This contradicts with the supposition that S is a strong resolving set. Thus, $\mid S \mid \geq$ n. □

Lemma 2.3 *For every integer $n \geq 3$ and n odd, a set $S = \{u_1, u_2, \ldots, u_n\}$ is a strong resolving set of antiprism graph A_n.*

Proof. For every integer $i, j \in [1,\ n]$ with $1 \leq i < j \leq n$, a vertex $u_i \in S$ which strongly resolves v_i dan v_j so that $v_j \in I[v_i, u_j]$. Thus, $S = \{u_1, u_2, \ldots, u_n\}$ is a strong resolving set of antiprism graph A_n. □

Lemma 2.4 *For every integer $n \geq 3$ and n even, if S is a strong resolving set of antiprism graph A_n then $\mid S \mid \geq$ n+1.*

Proof. We know that S is a strong resolving set of antiprism graph A_n. Suppose that S contains at most n vertices, then $\mid S \mid < n+1$. Let V_1, $V_2 \subset V(A_n)$, with $V_1 = \{u_1, u_2, \ldots, u_n\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$. Now, we define $S_1 = V_1 \cap S$ and $S_2 = V_2 \cap S$. Without loss of generality, we may take $\mid S_1 \mid = p$, $p \geq 0$ and $\mid S_2 \mid = q$, $q \geq 0$. Clearly $p + q \geq n+1$, if not then there are two distinct vertices v_a and v_b where $v_a \in V_1 \setminus S_1$ and $v_b \in V_2 \setminus S_2$ such that for every $s \in S$, we obtain $v_a \notin I[v_b,\ s]$ and $v_b \notin I[v_a,\ s]$. This contradicts with the supposition that S is a strong resolving set. Thus, $\mid S \mid \geq$ n+1. □

Lemma 2.5 *For every integer $n \geq 3$ and n even, a set $S = \{u_1, u_2, \ldots, u_{\frac{n}{2}}, u_{\frac{n}{2}+1}, v_1, v_2, \ldots, v_{\frac{n}{2}}\}$ is a strong resolving set of antiprism graph A_n.*

Proof. We prove that for every two distinct vertices $u, v \in V(A_n)\backslash S$, $u \neq v$ there exists a vertex $s \in S$ which strongly resolves u and v. There are three possible pairs of vertices.

(i) A pair of vertices (u_i, u_j) with $i, j = \frac{n}{2}+2, \frac{n}{2}+3, \ldots, n$, $i \neq j$.
For every integer $i, j \in [\frac{n}{2}+2, n]$ with $i < j$, we obtain the shortest u_i - u_1 path: $u_i, u_{i+1}, \ldots, u_j, \ldots, u_n, u_1$. Thus, $u_j \in I[u_i, u_1]$.

(ii) A pair of vertices (v_i, v_j) with $i, j = \frac{n}{2}+1, \frac{n}{2}+2, \ldots, n$, $i \neq j$.
For every integer $i, j \in [\frac{n}{2}+1, n]$ with $i < j$, we obtain the shortest v_i - v_1 path: $v_i, v_{i+1}, \ldots, v_j, \ldots, v_n, v_1$. Thus, $v_j \in I[v_i, v_1]$.

(iii) A pair of vertices (u_i, v_j) with $i = \frac{n}{2}+2, \frac{n}{2}+3, \ldots, n$ dan $j = \frac{n}{2}+1, \frac{n}{2}+2, \ldots, n$.
For every integer $i \in [\frac{n}{2}+2, n]$ and $j \in [\frac{n}{2}+1, n]$ with $i \leq j$, we obtain the shortest u_i - v_1 path: $u_i, v_i, \ldots, v_j, \ldots, v_n, v_1$. Thus, $v_j \in I[u_i, v_1]$. Then, for every integer $i \in [\frac{n}{2}+2, n]$ and $j \in [\frac{n}{2}+1, n]$ with $i > j$, we obtain the shortest u_i - $v_{\frac{n}{2}}$ path: $u_i, v_{i-1}, \ldots, v_j, \ldots, v_{\frac{n}{2}+1}, v_{\frac{n}{2}}$. Thus, $v_j \in I[u_i, v_{\frac{n}{2}}]$.

From every possible pairs of vertices, there exists a vertex $s \in S$ which strongly resolves u, $v \in V(K_{m,n}) \setminus S$. Thus, S is a strong resolving set of antiprism graph A_n. □

Theorem 2.1 *Let A_n be the antiprism graph with $n \geq 3$, then*

$$sdim(A_n) = \begin{cases} n, & n \text{ odd;} \\ n+1, & n \text{ even.} \end{cases}$$

Proof. Let A_n be an antiprism graph with $n \geq 3$ and a set of vertices $V(A_n) = \{u_1,\ u_2,\ \ldots,\ u_n,\ v_1,\ v_2,\ \ldots,\ v_n\}$. We divide the proof into two cases according to the values of n.

(i) For n odd.
By using Lemma 2.3, a set $S = \{u_1, u_2, \ldots, u_n\}$ is a strong resolving set of antiprism graph A_n. According to Lemma 2.2, $\mid S \mid \geq n$ so that S is a strong metric basis of antiprism graph A_n. Hence, $sdim(A_n) = n$.

(ii) For n even.
By using Lemma 2.5, a set $S = \{u_1, u_2, \ldots, u_{\frac{n}{2}+1}, v_1, v_2, \ldots, v_{\frac{n}{2}}\}$ is a strong resolving set of antiprism graph A_n. According to Lemma 2.4, $\mid S \mid \geq n$ so that S is a strong metric basis of antiprism graph A_n. Hence, $sdim(A_n) = n + 1$. □

2.3. The Strong Metric Dimension of King Graph

Weisstein [7] defined the king graph $K_{m,n}$ for $m, n \geq 2$ is the graph with mn vertices in which each vertex represents a square in an $m \times n$ chessboard, and each edge corresponds to a legal move by king.The king graph $K_{m,n}$ can be depicted as in Figure 2.

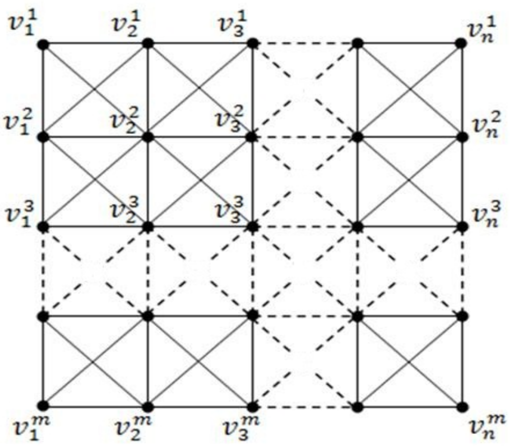

Figure 2. King graph $K_{m,n}$

Lemma 2.6 *For every integer m,n $\geq$ 2, if S is a strong resolving set of king graph $K_{m,n}$ then $\mid S \mid \geq$ m+n-1.*

Proof. We know that S is a strong resolving set of king graph $K_{m,n}$. Suppose that S contains at most $m + n - 2$ vertices, then $\mid S \mid < m + n - 1$. Let $V_1, V_2 \subset V(A_n)$, with $V_1 = \{v_1^1, v_2^1, \ldots, v_n^1, v_1^2, v_1^3, \ldots, v_1^m\}$ and $V_2 = \{v_2^2, v_3^2, \ldots, v_n^2, v_2^3, v_3^3, \ldots, v_n^3, \ldots, v_2^m, v_3^m, v_n^m\}$. Now, we define $S_1 = V_1 \cap S$ and $S_2 = V_2 \cap S$. Without loss of generality, we may take $\mid S_1 \mid = p$, $p > 0$ and $\mid S_2 \mid = q$, $q \geq 0$. Clearly $p + q \geq m + n - 1$, if not then there are two distinct vertices v_a and v_b where $v_a \in V_1 \setminus S_1$ and $v_b \in V_2 \setminus S_2$ such that for every $s \in S$, we obtain $v_a \notin I[v_b, s]$ and $v_b \notin I[v_a, s]$. This contradicts with the supposition that S is a strong resolving set. Thus, $\mid S \mid \geq$ m+n-1. □

Lemma 2.7 *For every integer m,n $\geq$ 2, a set $S = \{v_1^1, v_2^1, \ldots, v_n^1, v_1^2, v_1^3, \ldots, v_1^m\}$ is a strong resolving set of king graph $K_{m,n}$.*

Proof. We prove that for every two distinct vertices $u, v \in V(K_{m,n}) \backslash S$, $u \neq v$ there exists a vertex $s \in S$ which strongly resolves u and v. There are three possible pairs of vertices.

(i) A pair of vertices (v_k^p, v_l^p) with $p = 2, 3, \ldots, m$ and $k, l = 2, 3, \ldots, n$.
For every integer $p \in [2, m]$ and $k, l \in [2, n]$ with $k < l$, we obtain the shortest v_l^p - v_1^p path: $v_l^p, v_{l-1}^p, \ldots, v_k^p, \ldots, v_2^p, v_1^p$. Thus, $v_k^p \in I[v_l^p, v_1^p]$.

(ii) A pair of vertices (v_q^i, v_q^j) with $i, j = 2, 3, \ldots, m$ and $q = 2, 3, \ldots, n$.
For every integer $i, j \in [2, m]$, $q \in [2, n]$ with $i < j$, we obtain the shortest v_q^j - v_q^1 path: $v_q^j, v_q^{j-1}, \ldots, v_q^i, \ldots, v_q^2, v_q^1$. Thus, $v_q^i \in I[v_q^j, v_q^1]$.

(iii) A pair of vertices (v_k^i, v_l^j) with $i, j = 2, 3, \ldots, m$ and $k, l = 2, 3, \ldots, n$.
For every integer $i, j \in [2, m]$, $k, l \in [1, n]$ with $i < j$ dan $k < l$, we obtain the shortest v_l^j - v_k^1 path: $v_l^j, v_{l-1}^{j-1}, v_{l-1}^{j-1}, \ldots, v_k^{j-1}, v_k^{j-2}, \ldots, v_k^i, v_k^{i-1}, \ldots, v_k^2, v_k^1$. Thus, $v_k^i \in I[v_l^j, v_k^1]$.

From every possible pairs of vertices, there exists a vertex $s \in S$ which strongly resolves $u, v \in V(K_{m,n})\backslash S$. Thus S is a strong resolving set of king graph $K_{m,n}$. □

Theorem 2.2 *Let $K_{m,n}$ be the king graph with $m, n \geq 2$, then $sdim(K_{m,n}) = m + n - 1$.*

Proof. By using Lemma 2.7, a set $S = \{v_1^1, v_2^1, \ldots, v_n^1, v_1^2, v_1^3, \ldots, v_1^m\}$ is a strong resolving set of king graph $K_{m,n}$. According to Lemma 2.6, $\mid S \mid \geq m + n - 1$ so that S is a strong metric basis of king graph $K_{m,n}$. Hence, $sdim(K_{m,n}) = m + n - 1$. □

2.4. The Strong Metric Dimension of $K_m \odot K_n$ Graph

The corona product $K_m \odot K_n$ graph is graph obtained from K_m and K_n by taking one copy of K_m and n copies of K_n and joining by an edge each vertex from i^{th}-copy of K_n with the i^{th}-vertex of K_m. The $K_m \odot K_n$ can be depicted as in Figure 3.

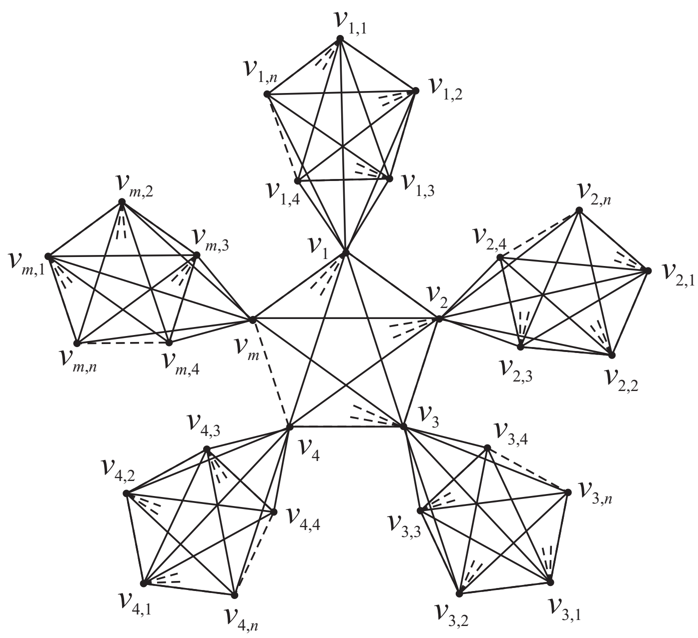

Figure 3. $K_m \odot K_n$ Graph

Lemma 2.8 *For every integer $m \geq 2$ and $n \geq 1$, if S is a strong resolving set of $K_m \odot K_n$ graph then $\mid S \mid \geq mn$-1.*

Proof. Let us consider a pair of vertices $(v_{i,k}, v_{j,l})$ for $i, j \in [1, m], k, l \in [1, n]$ satisfying both of the conditions of Lemma 2.1. According to Property 2.1, we obtain $v_{i,k} \in S$ or $v_{j,l} \in S$. It means that S contains one vertex from distinct sets $X = \{ v_{i,k}, v_{j,l} \}$ with $i, j = 1, 2, \ldots, m$ and $k, l = 1, 2, \ldots, n$. The minimum number of vertices from distinct sets X is $mn - 1$. Therefore, $\mid S \mid \geq mn - 1$. □

Lemma 2.9 *For every integer $m \geq 2$ and $n \geq 1$, a set $S = \{v_{1,1}, v_{1,2}, \ldots, v_{1,n-1}, v_{1,n}, v_{2,1}, v_{2,2}, \ldots, v_{2,n-1}, v_{2,n}, \ldots, v_{m,1}, v_{m,2}, \ldots, v_{m,n-1}\}$ is a strong resolving set of $K_m \odot K_n$ graph.*

Proof. For every integer $i \in [1, m-1]$, we obtain the shortest $v_{m,n}$ - $v_{i,k}$ path: $v_{m,n}, v_m, v_i, v_{i,k}$. So that $v_{i,k}$ strongly resolves a pair of vertices $(v_i, v_{m,n})$. Thus, $v_i \in I[v_{m,n}, v_{i,k}]$.

For a pair of vertices (v_i, v_j) with $i, j \in [1, m]$ and $i \neq j$, we obtain the shortest v_i - $v_{j,l}$ path: $v_i, v_j, v_{j,l}$. So that $v_{j,l}$ strongly resolves a pair of vertices (v_i, v_j). Thus, $v_j \in I[v_i, v_{j,l}]$.

Therefore $S = \{v_{1,1}, v_{1,2}, \ldots, v_{1,n-1}, v_{1,n}, v_{2,1}, v_{2,2}, \ldots, v_{2,n-1}, v_{2,n}, \ldots, v_{m,1}, v_{m,2}, \ldots, v_{m,n-1}\}$ is a strong resolving set of $K_m \odot K_n$ graph. □

Theorem 2.3 *Let $K_m \odot K_n$ graph with $m, n \geq 1$, then*

$$sdim(K_m \odot K_n) = \begin{cases} n, & m = 1,\ n \geq 1; \\ mn - 1, & m \geq 2, n \geq 1. \end{cases}$$

Proof. Suppose $K_m \odot K_n$ graph with $m, n, k \geq 1$ and a set of vertices $V(K_m \odot K_n)$=$\{v_1, v_2, \ldots, v_m, v_{1,1}, v_{1,2}, \ldots, v_{1,n}, v_{2,1}, v_{2,2}, \ldots, v_{2,n}, \ldots, v_{m,1}, v_{m,2}, \ldots, v_{m,n}\}$. We divide the proof into two cases according to the values of m and n.

(i) For $m = 1$ and $n \geq 1$.
By using characterization Yi [8], $sdim(G) = n-1$ if only if $G \cong K_n$, so that $sdim(K_1 \odot K_n) = n$ because $K_1 \odot K_n \cong K_{n+1}$ then strong resolving set of $K_1 \odot K_n$ graph has n element. Hence $sdim(K_1 \odot K_n) = n$.

(ii) For $m \geq 2$ and $n \geq 1$.
By using Lemma 2.9 a set $S = \{v_{1,1}, v_{1,2}, \ldots, v_{1,n-1}, v_{1,n}, v_{2,1}, v_{2,2}, \ldots, v_{2,n-1}, v_{2,n}, \ldots, v_{m,1}, v_{m,2}, \ldots, v_{m,n-1}\}$ is a strong resolving set of $K_m \odot K_n$ graph. According to Lemma 2.8, $\mid S \mid \geq mn-1$ so that S is a strong metric basis of $K_m \odot K_n$ graph. Hence $sdim(K_m \odot K_n) = mn-1$. □

3. Conclusion

According to the discussion above, it can be concluded that the strong metric dimension of an antiprism graph, a king graph, and a $K_m \odot K_n$ graph are as stated in Theorem 2.1, Theorem 2.2, and Theorem 2.3 respectively.

Open Problem: Determine the strong metric dimension of a $K_m \odot^k K_n$ graph.

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