On The Strong Metric Dimension of Sun Graph, Windmill Graph, and Möbius Ladder Graph

Mila Widyaningrum and Tri Atmojo Kusmayadi
Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Sebelas Maret, Surakarta, Indonesia
E-mail: milawidyaningrum4@gmail.com, tri.atmojo.kusmayadi@gmail.com

Abstract. Let $G$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. The interval $I[u,v]$ between $u$ and $v$ is defined as the collection of all vertices that belong to some shortest $u$-$v$ path. A vertex $s \in V(G)$ strongly resolves two vertices $u$ and $v$ if $u$ belongs to a shortest $v$-$s$ path or $v$ belongs to a shortest $u$-$s$ path. A set $S \subset V(G)$ is a strong resolving set if every two distinct vertices of $G$ are strongly resolved by some vertex of $S$. The smallest cardinality of strong resolving set is called a strong metric basis. The strong metric dimension of $G$, denoted by $sdim(G)$, is defined as the cardinality of the strong metric basis. In this paper we determine the strong metric dimension of a sun graph $S_n$, a windmill graph $K_{m,n}$, and a Möbius ladder graph $M_n$. We obtain the strong metric dimension of sun graph $S_n$ is $n - 1$ for $n \geq 3$. The strong metric dimension of windmill graph $K_{m,n}$ is $(n - 1)m - 1$ for $m \geq 2$ and $n \geq 3$. The strong metric dimension of Möbius ladder graph $M_n$ with $n \geq 5$ is $2\lceil \frac{n+2}{4} \rceil$ for $n$ even.

1. Introduction

Sebő and Tannier [8] introduced the concept of strong metric dimension in 2004. In 2007 the strong metric dimension of graph and digraph was introduced by Oellermann and Peters-Fransen [7]. Let $G$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$, Oellermann and Peters-Fransen [7] defined for two vertices $u, v \in V(G)$, the interval $I[u,v]$ between $u$ and $v$ to be the collection of all vertices that belong to some shortest $u$-$v$ path. If $u$ belongs to a shortest $v$-$s$ path, denoted by $u \in I[v,s]$, or $v$ belongs to a shortest $u$-$s$ path, denoted by $v \in I[u,s]$ then a vertex $s$ strongly resolves two vertices $u$ and $v$. A set of vertices $S \subset V(G)$ is a strong resolving set for $G$ if every two distinct vertices of $G$ are strongly resolved by some vertex of $S$. The smallest cardinality of strong resolving set is called a strong metric basis. The cardinality of the strong metric basis is the strong metric dimension of $G$, denoted by $sdim(G)$.

Some authors have investigated the strong metric dimension to some graph classes. In 2004 Sebő and Tannier [8] observed the strong metric dimension of complete graph $K_n$, cycle graph $C_n$, and tree. Kratica [3] determined the strong metric dimension of hamming graph $H_{n,k}$ in 2012. At the same year, Kratica [4] already observed the strong metric dimension of some convex polytope graph. In 2013 Yi [9] determined that the metric dimension of $G$ is 1 if and only if $G$ is a path $P_n$ and the strong metric dimension of $G$ is $n - 1$ if and only if $G$ is complete graph $K_n$. Kusmayadi et al. [5] determined the strong metric dimension of some related wheel graph such as sunflower graph, $t$-fold wheel graph, helm graph, and friendship graph in 2016. In this paper we determine the strong metric dimension of a sun graph $S_n$, a windmill graph $K_{m,n}$, and a Möbius Ladder graph $M_n$ for $n$ even.
2. Main Results

2.1. Strong Metric Dimension

Let $G$ be a connected graph with a set of vertices $V(G)$ and a set of edges $E(G)$. Let $S$ be a subset of $V(G)$. Oellermann and Peters-Fransen [7] defined for two vertices $u, v \in V(G)$, the interval $I[u, v]$ between $u$ and $v$ to be the collection of all vertices that belong to some shortest $u - v$ path. A vertex $s \in S$ strongly resolves two vertices $u, v$ if $u \in I[v, s]$ or $v \in I[u, s]$. A vertex set $S$ of $G$ is a strong resolving set of $G$ if every two distinct vertices of $G$ are strongly resolved by some vertex of $S$. The smallest cardinality of strong resolving set is called strong metric basis of $G$. The strong metric dimension of a graph $G$ is defined as the cardinality of strong metric basis denoted by $sdim(G)$. We often make use of the following lemma and properties about strong metric dimension given by Kratica et al. [4].

**Lemma 2.1** Let $u, v \in V(G)$, $u \neq v$,

(i) $d(w,v) \leq d(u,v)$ for each $w$ such that $uw \in E(G)$, and
(ii) $d(u,w) \leq d(u,v)$ for each $w$ such that $vw \in E(G)$.

Then there does not exist vertex $a \in V(G)$, $a \neq u,v$ that strongly resolves vertices $u$ and $v$.

**Property 2.1** If $S \subset V(G)$ is strong resolving set of graph $G$, then for every two vertices $u, v \in V(G)$ satisfying conditions 1 and 2 of Lemma 2.1, obtained $u \in S$ or $v \in S$.

**Property 2.2** If $S \subset V(G)$ is strong resolving set of graph $G$, then for every two vertices $u, v \in V(G)$ satisfying $d(u,v) = \text{diam}(G)$, obtained $u \in S$ or $v \in S$.

2.2. The Strong Metric Dimension of Sun Graph

Mirzakhah and Kiani [6] defined the sun graph $S_n$ for $n \geq 3$ as a graph obtained by connecting each vertex of cycle $C_n$ to a pendant vertex with an edge. The sun graph $S_n$ can be depicted as in Figure 1.

![Sun graph $S_n$](image)

**Figure 1.** Sun graph $S_n$

**Theorem 2.1** Let $S_n$ be the sun graph with $n \geq 3$. Then $sdim(S_n) = n - 1$.

**Proof.** First, we prove that for every integer $n \geq 3$, if $S$ is a strong resolving set of sun graph $S_n$ then $|S| \geq n - 1$. We know that $S$ is a strong resolving set of sun graph $S_n$. Suppose that $S$ contains at most $n - 2$ vertices, then $|S| < n - 1$. Let $V_1, V_2 \subset V(S_n)$, with $V_1 = \{v_1, v_2, \ldots, v_n\}$ and $V_2 = \{u_1, u_2, \ldots, u_n\}$. Now, we define $S_1 = V_1 \cap S$ and $S_2 = V_2 \cap S$. Without loss of generality, we may take $|S_1| = a, a > 0$ and $|S_2| = b, b \geq 0$. We obtain $a + b \leq n - 2$, there...
are two distinct vertices \( v_i \) and \( v_j \in V_i \setminus S_1 \), where \( i \neq j \) such that for every \( s \in S \), \( v_i \notin I[v_j, s] \) and \( v_j \notin I[v_i, s] \). This contradicts with the supposition that \( S \) is a strong resolving set. Thus \( |S| \geq n - 1 \).

Then, we prove that for every integer \( n \geq 3 \), if \( S = \{v_1, v_2, \ldots, v_{n-1}\} \) then \( S \) is a strong resolving set of sun graph \( S_n \). We prove that for every two distinct vertices \( u, v \in V(S_n) \setminus S \) where \( u \neq v \) is strongly resolved by a vertex \( s \in S \). There are two possible pairs of vertices.

(i) A pair of vertices \((u_i, v_n)\) with \( i = 1, 2, \ldots, n \).

For every integer \( 1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor \), we obtain the shortest path between \( v_{\lfloor \frac{n-1}{2} \rfloor}, u_{\lfloor \frac{n-1}{2} \rfloor}, u_{\lfloor \frac{n-1}{2} \rfloor} - 1, \ldots, u_1, u_{n-1}, v_n \). So that \( u_i \in I[v_{\lfloor \frac{n-1}{2} \rfloor}, v_n] \). Then for every integer \( \lfloor \frac{n+1}{2} \rfloor \leq i \leq n \), we obtain the shortest path between \( v_{\lfloor \frac{n+1}{2} \rfloor}, u_{\lfloor \frac{n+1}{2} \rfloor}, u_{\lfloor \frac{n+1}{2} \rfloor} + 1, \ldots, u_{n-1}, u_n, v_n \). So that \( u_i \in I[v_{\lfloor \frac{n+1}{2} \rfloor}, v_n] \).

(ii) A pair of vertices \((u_i, u_j)\) with \( i, j = 1, 2, \ldots, n \) and \( i \neq j \).

For every integer \( i, j \in [1, n] \) without loss of generality we may assume \( 1 \leq i < j \leq n \), therefore \( u_i \) and \( u_j \) will be strongly resolved by \( v_i \) so that \( u_i \in I[v_i, u_j] \).

From every possible pairs of vertices, there exists a vertex \( v_i \in S \) with \( i \in [1, n-1] \) which strongly resolves two distinct vertices of \( V(S_n) \setminus S \). Thus \( S \) is a strong resolving set of sun graph \( S_n \).

We obtain a set \( S = \{v_1, v_2, \ldots, v_{n-1}\} \) is a strong resolving set of sun graph \( S_n \) with \( n \geq 3 \) and we have \( |S| \geq n - 1 \), so that \( S \) is a strong metric basis of sun graph \( S_n \). Hence \( sdim(S_n) = n - 1 \).

\[ \Box \]

2.3. The Strong Metric Dimension of Windmill Graph

Gallian [1] defined the windmill graph \( K_n^m \) is a graph consists of \( m \) copies of \( K_n \) with a vertex in common. The windmill graph \( K_n^m \) can be depicted as in Figure 2.

![Windmill graph \( K_n^m \)](image)

**Figure 2. Windmill graph \( K_n^m \)**

**Theorem 2.2** Let \( K_n^m \) be the windmill graph with \( m \geq 2 \) and \( n \geq 3 \). Then \( sdim(K_n^m) = (n-1)m-1 \).

**Proof.** First, we prove that for every integer \( m \geq 2 \) and \( n \geq 3 \), if \( S \) is a strong resolving set of windmill graph \( K_n^m \) then \( |S| \geq (n-1)m - 1 \). Consider a pair of vertices \((v_i, v_j)\) for \( i, j = 1, 2, \ldots, (n-1)m \) and \( i \neq j \) satisfying conditions 1 and 2 of Lemma 2.1. According to Property 2.1,
we obtain $v_i \in S$ or $v_j \in S$. It means $S$ has at least one vertex from sets $X_{ij} = \{v_i, v_j\}$ with $i, j = 1, 2, \ldots, (n-1)m$ and $i \neq j$. The minimum number of vertices from sets $X_{ij}$ is $(n-1)m - 1$. Therefore, $|S| \geq (n-1)m - 1$.

After that, we prove that for every integer $m \geq 2$ and $n \geq 3$, if $S = \{v_1, v_2, \ldots, v_{(n-1)m-1}\}$ then $S$ is a strong resolving set of $K_n^m$ graph. For every two distinct vertices $u, v \in V(K_n^m) \setminus S$, there is a vertex $s \in S$ which strongly resolves $u$ and $v$. Let us prove that a pair of vertices $(c, v_{(n-1)m})$ is strongly resolved by $v_i$ for $i = 1, 2, \ldots, (n-1)(m-1)$. The shortest path between $v_{(n-1)m}$ and $v_i$ is $v_{(n-1)m}, c, v_i$. Since vertex $c$ belongs to this shortest path, $v_i$ strongly resolves $c$ and $v_{(n-1)m}$. Thus $S$ is a strong resolving set of windmill graph $K_n^m$.

We have a set $S = \{v_1, v_2, \ldots, v_{(n-1)m-1}\}$ is a strong resolving set of windmill graph $K_n^m$ and $|S| \geq (n-1)m - 1$, so that $S$ is a strong metric basis of windmill graph $K_n^m$. Hence $sdim(K_n^m) = (n-1)m - 1$. □

2.4. The Strong Metric Dimension of Möbius Ladder Graph

Guy and Harary [2] defined the Möbius ladder graph $M_n$ is a graph constructed by connecting vertices $u$ and $v$ in the cycle $C_n$ if $d(u, v) = \text{diam}(C_n)$. The Möbius ladder graph $M_n$ can be depicted as in Figure 3 and Figure 4.

![Figure 3. Möbius ladder graph $M_n$ for $n$ even](image)

![Figure 4. Möbius ladder graph $M_n$ for $n$ odd](image)

**Theorem 2.3** Let $M_n$ be the Möbius ladder graph with $n \geq 5$. Then $sdim(M_n) = 2 \left\lceil \frac{n+2}{4} \right\rceil$ for $n$ even.

**Proof.** First, we prove that for every $n \geq 5$, if $S$ is a strong resolving set of Möbius ladder graph $M_n$ then $|S| \geq 2 \left\lceil \frac{n+2}{4} \right\rceil$ for $n$ even. We know that $S$ is a strong resolving set of Möbius ladder
graph $M_n$. Suppose that $S$ contains at most $2 \lceil \frac{n+2}{4} \rceil - 1$ vertices, then $|S| < 2 \lceil \frac{n+2}{4} \rceil$. Let $V_1, V_2 \subset V(M_n)$, with $V_1 = \{v_1, v_2, \ldots, v_{\frac{n}{2}}\}$ and $V_2 = \{v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, \ldots, v_n\}$. Now, we define $S_1 = V_1 \cap S$ and $S_2 = V_2 \cap S$. Without loss of generality, we may take $|S_1| = p$, $p > 0$ and $|S_2| = q$, $q > 0$. Clearly $p + q \geq 2 \lceil \frac{n+2}{4} \rceil$. Otherwise, there are two distinct vertices $v_i$ and $v_j$, where $v_i \in V_1 \setminus S_1$ and $v_j \in V_2 \setminus S_2$ such that for every $s \in S$ we obtain $v_i \notin I[v_j, s]$ and $v_j \notin I[v_i, s]$. This contradicts with the supposition that $S$ is a strong resolving set. Thus $|S| \geq 2 \lceil \frac{n+2}{4} \rceil$.

Then, we prove that for every even integer $n \geq 5$, a set $S = \{v_1, v_2, \ldots, v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, \ldots, v_{\frac{n}{2}+\lceil \frac{n+2}{4} \rceil}\}$ is a strong resolving set of Möbius ladder graph $M_n$ for $n$ even. We prove that for every two distinct vertices $v_i, v_j \in V(M_n) \setminus S$, where $i, j = (\lceil \frac{n+2}{4} \rceil, \frac{n}{2}) \cup (\frac{n}{2} + \lceil \frac{n+2}{4} \rceil, n)$ is strongly resolved by a vertex $s \in S$. There are three possible pairs of vertices.

(i) A pair of vertices $(v_i, v_j)$ where $\lceil \frac{n+2}{4} \rceil < i, j < \frac{n}{2}$, $i \neq j$.

For every integer $\lceil \frac{n+2}{4} \rceil < i, j < \frac{n}{2}$ with $\lceil \frac{n+2}{4} \rceil < i < j < \frac{n}{2}$, we obtain the shortest path between $v_{\lceil \frac{n+2}{4} \rceil}$ and $v_j$ is $v_{\lceil \frac{n+2}{4} \rceil}, v_{\lceil \frac{n+2}{4} \rceil}+1, \ldots, v_i, \ldots, v_j$. So that $v_i \in I[v_{\lceil \frac{n+2}{4} \rceil}, v_j]$.

(ii) A pair of vertices $(v_i, v_j)$ where $\frac{n}{2} + \lceil \frac{n+2}{4} \rceil < i, j \leq n$, $i \neq j$.

For every integer $\frac{n}{2} + \lceil \frac{n+2}{4} \rceil < i, j \leq n$ with $\frac{n}{2} + \lceil \frac{n+2}{4} \rceil < i < j \leq n$, we obtain the shortest path between $v_{\frac{n}{2} + \lceil \frac{n+2}{4} \rceil}$ and $v_j$ is $v_{\frac{n}{2} + \lceil \frac{n+2}{4} \rceil}, v_{\frac{n}{2} + \lceil \frac{n+2}{4} \rceil}+1, \ldots, v_i, \ldots, v_j$. So that $v_i \in I[v_{\frac{n}{2} + \lceil \frac{n+2}{4} \rceil}, v_j]$.

(iii) A pair of vertices $(v_i, v_j)$ where $\lceil \frac{n+2}{4} \rceil < i \leq \frac{n}{2}$ and $\frac{n}{2} + \lceil \frac{n+2}{4} \rceil < j \leq n$.

For every pair of vertices $(v_i, v_j)$ with $\lceil \frac{n+2}{4} \rceil < i < \frac{n}{2}$ and $\frac{n}{2} + \lceil \frac{n+2}{4} \rceil < j < n$ with $\frac{n}{2} + \lceil \frac{n+2}{4} \rceil < j \leq i + \frac{n}{2}$, we obtain the shortest path between $v_i$ and $v_j$ is $v_i, v_i+\frac{n}{2}, v_i+\frac{n}{2}+1, v_i+\frac{n}{2}+2, \ldots, v_{\frac{n}{2} + \lceil \frac{n+2}{4} \rceil}$. So that $v_j \in I[v_i, v_{\frac{n}{2} + \lceil \frac{n+2}{4} \rceil}]$. If $i + \frac{n}{2} < j \leq n$, then $v_i$ and $v_j$ will be strongly resolved by $v_1$ with the shortest path between $v_i$ and $v_j$ is $v_i, v_i+\frac{n}{2}, v_i+\frac{n}{2}+1, \ldots, v_i, \ldots, v_j$. So that $v_j \in I[v_i, v_j]$.

From every possible pairs of vertices, there exists a vertex $s \in S$ which strongly resolves two distinct vertices of $V(M_n) \setminus S$. Thus $S$ is a strong resolving set of Möbius ladder graph $M_n$.

We obtain a set $S = \{v_1, v_2, \ldots, v_{\lceil \frac{n+2}{4} \rceil}, v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, \ldots, v_{\frac{n}{2}+\lceil \frac{n+2}{4} \rceil}\}$ is a strong resolving set of Möbius ladder graph $M_n$ and we have $|S| \geq 2 \lceil \frac{n+2}{4} \rceil$, so that $S$ is a strong metric basis of $M_n$. Hence $sdim(M_n) = 2 \lceil \frac{n+2}{4} \rceil$ for $n$ even.

\[\square\]

3. Conclusion

According to the discussion above, we have the strong metric dimension of a sun graph $S_n$, a windmill graph $K_n^m$, and a Möbius ladder graph $M_n$ for $n$ even. Furthermore, we conclude this paper with the following open problem for the direction of further research which is still in progress.

**Open Problem:** Determine the strong metric dimension of a Möbius ladder graph $M_n$ for $n$ odd.

Acknowledgments

The authors gratefully acknowledge the support from Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Sebelas Maret, Surakarta. Then, we wish to thank the referees for their suggestions which helped to improve the paper.

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