On The Strong Metric Dimension of Sun Graph, Windmill Graph, and Möbius Ladder Graph

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Abstract. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. The interval $I[u,v]$ between u and v is defined as the collection of all vertices that belong to some shortest u-v path. A vertex $s \in V(G)$ strongly resolves two vertices u and v if u belongs to a shortest v-s path or v belongs to a shortest u-s path. A set $S \subset V(G)$ is a strong resolving set if every two distinct vertices of G are strongly resolved by some vertex of S. The smallest cardinality of strong resolving set is called a strong metric basis. The strong metric dimension of G, denoted by $sdim(G)$, is defined as the cardinality of the strong metric basis. In this paper we determine the strong metric dimension of a sun graph S_n, a windmill graph K_n^m, and a Möbius ladder graph M_n. We obtain the strong metric dimension of sun graph S_n is n - 1 for $n \geq 3$. The strong metric dimension of windmill graph K_n^m is $(n$ - $1)m$ - 1 for $m \geq 2$ and $n \geq 3$. The strong metric dimension of Möbius ladder graph M_n with $n \geq 5$ is $2\lceil \frac{n+2}{4} \rceil$ for n even.

1. Introduction

Sebö and Tannier [8] introduced the concept of strong metric dimension in 2004. In 2007 the strong metric dimension of graph and digraph was introduced by Oellermann and Peters-Fransen [7]. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$, Oellermann and Peters-Fransen [7] defined for two vertices $u, v \in V(G)$, the interval $I[u,v]$ between u and v to be the collection of all vertices that belong to some shortest u - v path. If u belongs to a shortest v-s path, denoted by $u \in I[v,s]$, or v belongs to a shortest u-s path, denoted by $v \in I[u,s]$ then a vertex s strongly resolves two vertices u and v. A set of vertices $S \subset V(G)$ is strong resolving set for G if every two distinct vertices of G are strongly resolved by some vertex of S. The smallest cardinality of strong resolving set is called strong metric basis. The cardinality of the strong metric basis is a strong metric dimension of G, denoted by $sdim(G)$.

Some authors have investigated the strong metric dimension to some graph classes. In 2004 Sebö and Tannier [8] observed the strong metric dimension of complete graph K_n, cycle graph C_n, and tree. Kratica [3] determined the strong metric dimension of hamming graph $H_{n,k}$ in 2012. At the same year, Kratica [4] already observed the strong metric dimension of some convex polytope graph. In 2013 Yi [9] determined that the metric dimension of G is 1 if and only if G is a path P_n and the strong metric dimension of G is n - 1 if and only if G is complete graph K_n. Kusmayadi et al. [5] determined the strong metric dimension of some related wheel graph such as sunflower graph, t-fold wheel graph, helm graph, and friendship graph in 2016. In this paper we determine the strong metric dimension of a sun graph S_n, a windmill graph K_n^m, and a Möbius Ladder graph M_n for n even.

2. Main Results

2.1. Strong Metric Dimension

Let G be a connected graph with a set of vertices $V(G)$ and a set of edges $E(G)$. Let S be a subset of $V(G)$. Oellermann and Peters-Fransen [7] defined for two vertices $u, v \in V(G)$, the interval $I[u,v]$ between u and v to be the collection of all vertices that belong to some shortest $u - v$ path. A vertex $s \in S$ strongly resolves two vertices u and v if $u \in I[v,s]$ or $v \in I[u,s]$. A vertex set S of G is a strong resolving set of G if every two distinct vertices of G are strongly resolved by some vertex of S. The smallest cardinality of strong resolving set is called strong metric basis of G. The strong metric dimension of a graph G is defined as the cardinality of strong metric basis denoted by $sdim(G)$. We often make use of the following lemma and properties about strong metric dimension given by Kratica et al. [4].

Lemma 2.1 *Let u, v* $\in$ *V(G), u* $\neq$ *v,*

(i) d(w,v) $\leq$ *d(u,v) for each w such that uw* $\in$ *E(G), and*

(ii) d(u,w) $\leq$ *d(u,v) for each w such that vw* $\in$ *E(G).*

Then there does not exist vertex a $\in$ *V(G), a* $\neq$ *u,v that strongly resolves vertices u and v.*

Property 2.1 *If* $S \subset V(G)$ *is strong resolving set of graph G, then for every two vertices* $u, v \in V(G)$ *satisfying conditions* 1 *and* 2 *of Lemma* 2.1*, obtained* $u \in S$ *or* $v \in S$.

Property 2.2 *If* $S \subset V(G)$ *is strong resolving set of graph G, then for every two vertices* $u, v \in V(G)$ *satisfying d(u,v) = diam(G), obtained* $u \in S$ *or* $v \in S$.

2.2. The Strong Metric Dimension of Sun Graph

Mirzakhah and Kiani [6] defined the sun graph S_n for $n \geq 3$ as a graph obtained by connecting each vertex of cycle C_n to a pendant vertex with an edge. The sun graph S_n can be depicted as in Figure 1.

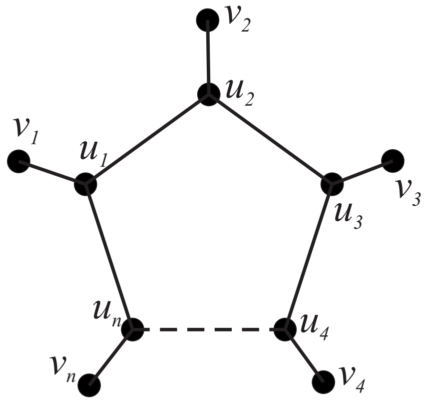


Figure 1. Sun graph S_n

Theorem 2.1 *Let* S_n *be the sun graph with* $n \geq 3$*. Then* $sdim(S_n) = n - 1$.

Proof. First, we prove that for every integer $n \geq 3$, if S is a strong resolving set of sun graph S_n then $\mid S \mid \geq n - 1$. We know that S is a strong resolving set of sun graph S_n. Suppose that S contains at most n - 2 vertices, then $\mid S \mid < n$ - 1. Let $V_1, V_2 \subset V(S_n)$, with $V_1 = \{v_1, v_2, \ldots, v_n\}$ and $V_2 = \{u_1, u_2, \ldots, u_n\}$. Now, we define $S_1 = V_1 \cap S$ and $S_2 = V_2 \cap S$. Without loss of generality, we may take $\mid S_1 \mid = a$, $a > 0$ and $\mid S_2 \mid = b$, $b \geq 0$. We obtain $a + b \leq n$ - 2, there are two distinct vertices v_i and $v_j \in V_1 \setminus S_1$, where $i \neq j$ such that for every $s \in S$, $v_i \notin I[v_j, s]$ and $v_j \notin I[v_i, s]$. This contradicts with the supposition that S is a strong resolving set. Thus $|\ S\ | \geq n$ - 1.

Then, we prove that for every integer $n \geq 3$, if $S = \{v_1, v_2, \ldots, v_{n-1}\}$ then S is a strong resolving set of sun graph S_n. We prove that for every two distinct vertices $u, v \in V(S_n) \setminus S$ where $u \neq v$ is strongly resolved by a vertex $s \in S$. There are two possible pairs of vertices.

(i) A pair of vertices (u_i, v_n) with i = 1, 2, ..., n.
For every integer $1 \leq i \leq \lceil\frac{n-1}{2}\rceil$, we obtain the shortest path between $v_{\lceil\frac{n-1}{2}\rceil}$ and v_n is $v_{\lceil\frac{n-1}{2}\rceil}, u_{\lceil\frac{n-1}{2}\rceil}, u_{\lceil\frac{n-1}{2}\rceil-1}, \ldots, u_1, u_n, v_n$. So that $u_i \in I[v_{\lceil\frac{n-1}{2}\rceil}, v_n]$. Then for every integer $\lfloor\frac{n+1}{2}\rfloor \leq i \leq n$, we obtain the shortest path between $v_{\lfloor\frac{n+1}{2}\rfloor}$ and v_n is $v_{\lfloor\frac{n+1}{2}\rfloor}, u_{\lfloor\frac{n+1}{2}\rfloor}, u_{\lfloor\frac{n+1}{2}\rfloor+1}, \ldots, u_{n-1}, u_n, v_n$. So that $u_i \in I[v_{\lfloor\frac{n+1}{2}\rfloor}, v_n]$.

(ii) A pair of vertices (u_i, u_j) with i, j = 1, 2, ..., n and $i \neq j$.
For every integer $i, j \in$ [1, n] without loss of generality we may assume $1 \leq i < j \leq n$, therefore u_i and u_j will be strongly resolved by v_i so that $u_i \in I[v_i, u_j]$.

From every possible pairs of vertices, there exists a vertex $v_i \in S$ with $i \in [1,\ n-1]$ which strongly resolves two distinct vertices of $V(S_n) \setminus S$. Thus S is a strong resolving set of sun graph S_n.

We obtain a set $S = \{v_1, v_2, \ldots, v_{n-1}\}$ is a strong resolving set of sun graph S_n with $n \geq 3$ and we have $|\ S\ | \geq n$ - 1, so that S is a strong metric basis of sun graph S_n. Hence $sdim(S_n)$ = n - 1. □

2.3. The Strong Metric Dimension of Windmill Graph

Gallian [1] defined the windmill graph K_n^m is a graph consists of m copies of K_n with a vertex in common. The windmill graph K_n^m can be depicted as in Figure 2.

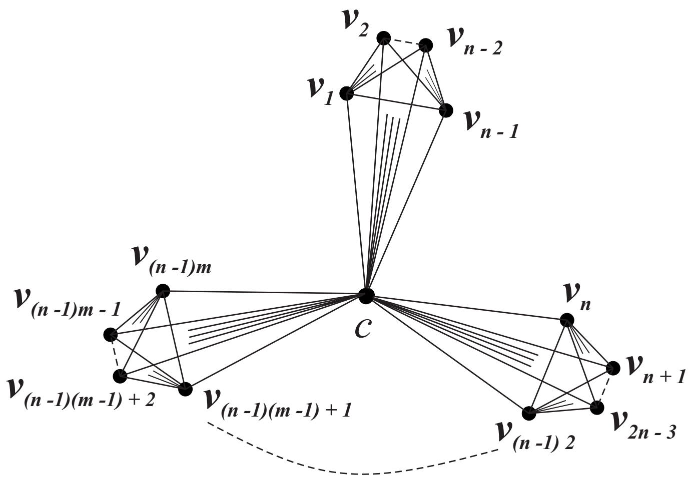


Figure 2. Windmill graph K_n^m

Theorem 2.2 *Let K_n^m be the windmill graph with $m \geq 2$ and $n \geq 3$. Then $sdim(K_n^m) = (n-1)m - 1$.*

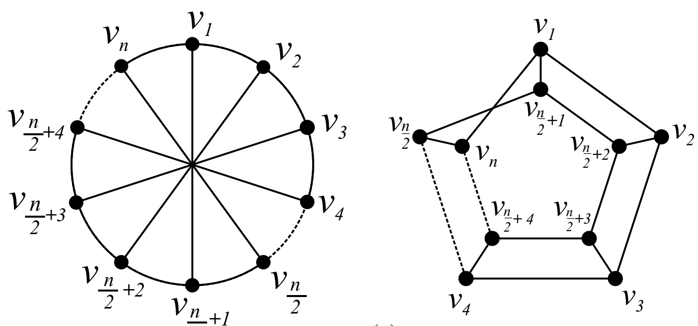
Proof. First, we prove that for every integer $m \geq 2$ and $n \geq 3$, if S is a strong resolving set of windmill graph K_n^m then $|\ S\ | \geq (n$ - 1$)m$ - 1. Consider a pair of vertices (v_i, v_j) for i, j = 1, 2, ..., $(n-1)m$ and $i \neq j$ satisfying conditions 1 and 2 of Lemma 2.1. According to Property 2.1, we obtain $v_i \in S$ or $v_j \in S$. It means S has at least one vertex from sets $X_{ij} = \{v_i, v_j\}$ with i, j = 1, 2, ..., $(n-1)m$ and $i \neq j$. The minimum number of vertices from sets X_{ij} is $(n$ - 1$)m$ - 1. Therefore, $\mid S \mid \geq (n-1)m{-}1$.

After that, we prove that for every integer $m \geq 2$ and $n \geq 3$, if $S = \{v_1, v_2, \ldots, v_{(n-1)m-1}\}$ then S is a strong resolving set of K_n^m graph. For every two distinct vertices $u, v \in V(K_n^m)\backslash S$, there is a vertex $s \in S$ which strongly resolves u and v. Let us prove that a pair of vertices $(c, v_{(n-1)m})$ is strongly resolved by v_i for i = 1, 2, ..., $(n-1)(m-1)$. The shortest path between $v_{(n-1)m}$ and v_i is $v_{(n-1)m}$, c, v_i. Since vertex c belongs to this shortest path, v_i strongly resolves c and $v_{(n-1)m}$. Thus S is a strong resolving set of windmill graph K_n^m.

We have a set $S = \{v_1, v_2, \ldots, v_{(n-1)m-1}\}$ is a strong resolving set of windmill graph K_n^m and $|S| \geq (n$ - 1$)m$ - 1, so that S is a strong metric basis of windmill graph K_n^m. Hence $sdim(K_n^m) =$ $(n$ - 1$)m$ - 1. $\square$

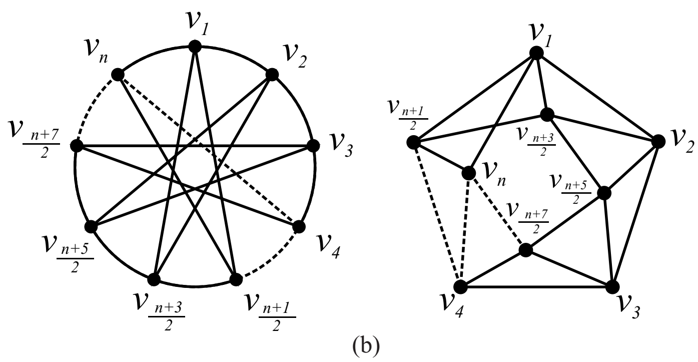
2.4. The Strong Metric Dimension of Möbius Ladder Graph

Guy and Harary [2] defined the Möbius ladder graph M_n is a graph constructed by connecting vertices u and v in the cycle C_n if $d(u,v)$ = $diam(C_n)$. The Möbius ladder graph M_n can be depicted as in Figure 3 and Figure 4.



(a)

Figure 3. Möbius ladder graph M_n for n even



(b)

Figure 4. Möbius ladder graph M_n for n odd

Theorem 2.3 *Let M_n be the Möbius ladder graph with $n \geq 5$. Then sdim(M_n) = 2 $\lceil \frac{n+2}{4} \rceil$ for n even.*

Proof. First, we prove that for every $n \geq 5$, if S is a strong resolving set of Möbius ladder graph M_n then $\mid S \mid \geq 2 \lceil \frac{n+2}{4} \rceil$ for n even. We know that S is a strong resolving set of Möbius ladder graph M_n. Suppose that S contains at most $2\left\lceil\frac{n+2}{4}\right\rceil - 1$ vertices, then $|S| < 2\left\lceil\frac{n+2}{4}\right\rceil$. Let $V_1, V_2 \subset V(M_n)$, with $V_1 = \{v_1, v_2, \ldots, v_{\frac{n}{2}}\}$ and $V_2 = \{v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, \ldots, v_n\}$. Now, we define $S_1 = V_1 \cap S$ and $S_2 = V_2 \cap S$. Without loss of generality, we may take $|S_1| = p$, $p > 0$ and $|S_2| = q$, $q > 0$. Clearly $p + q \geq 2\left\lceil\frac{n+2}{4}\right\rceil$. Otherwise, there are two distinct vertices v_i and v_j, where $v_i \in V_1 \backslash S_1$ and $v_j \in V_2 \backslash S_2$ such that for every $s \in S$ we obtain $v_i \notin I[v_j, s]$ and $v_j \notin I[v_i, s]$. This contradicts with the supposition that S is a strong resolving set. Thus $|S| \geq 2\left\lceil\frac{n+2}{4}\right\rceil$.

Then, we prove that for every even integer $n \geq 5$, a set $S = \{v_1, v_2, \ldots, v_{\lceil\frac{n+2}{4}\rceil}, v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, \ldots, v_{\frac{n}{2}+\lceil\frac{n+2}{4}\rceil}\}$ is a strong resolving set of Möbius ladder graph M_n for n even. We prove that for every two distinct vertices $v_i, v_j \in V(M_n)\backslash S$, where $i, j = (\lceil\frac{n+2}{4}\rceil, \frac{n}{2}] \cup (\frac{n}{2}+\lceil\frac{n+2}{4}\rceil, n]$ is strongly resolved by a vertex $s \in S$. There are three possible pairs of vertices.

(i) A pair of vertices (v_i, v_j) where $\lceil\frac{n+2}{4}\rceil < i, j \leq \frac{n}{2}$, $i \neq j$.
For every integer $\lceil\frac{n+2}{4}\rceil < i, j \leq \frac{n}{2}$ with $\lceil\frac{n+2}{4}\rceil < i < j \leq \frac{n}{2}$, we obtain the shortest path between $v_{\lceil\frac{n+2}{4}\rceil}$ and v_j is $v_{\lceil\frac{n+2}{4}\rceil}, v_{\lceil\frac{n+2}{4}\rceil+1}, \ldots, v_i, \ldots, v_j$. So that $v_i \in I[v_{\lceil\frac{n+2}{4}\rceil}, v_j]$.

(ii) A pair of vertices (v_i, v_j) where $\frac{n}{2}+\lceil\frac{n+2}{4}\rceil < i, j \leq n$, $i \neq j$.
For every integer $\frac{n}{2}+\lceil\frac{n+2}{4}\rceil < i, j \leq n$ with $\frac{n}{2}+\lceil\frac{n+2}{4}\rceil < i < j \leq n$, we obtain the shortest path between $v_{\frac{n}{2}+\lceil\frac{n+2}{4}\rceil}$ and v_j is $v_{\frac{n}{2}+\lceil\frac{n+2}{4}\rceil}, v_{\frac{n}{2}+\lceil\frac{n+2}{4}\rceil+1}, \ldots, v_i, \ldots, v_j$. So that $v_i \in I[v_{\frac{n}{2}+\lceil\frac{n+2}{4}\rceil}, v_j]$.

(iii) A pair of vertices (v_i, v_j) where $\lceil\frac{n+2}{4}\rceil < i \leq \frac{n}{2}$ and $\frac{n}{2} + \lceil\frac{n+2}{4}\rceil < j \leq n$.
For every pair of vertices (v_i, v_j) with $\lceil\frac{n+2}{4}\rceil < i \leq \frac{n}{2}$ and $\frac{n}{2} + \lceil\frac{n+2}{4}\rceil < j \leq n$. Vertex v_i and v_j will be strongly resolved by a vertex $v_{\frac{n}{2}+\lceil\frac{n+2}{4}\rceil}$ if $\frac{n}{2} + \lceil\frac{n+2}{4}\rceil < j \leq i + \frac{n}{2}$, we obtain the shortest path between v_i and $v_{\frac{n}{2}+\lceil\frac{n+2}{4}\rceil}$ is $v_i, v_{i+\frac{n}{2}}, v_{i+\frac{n}{2}-1}, v_{i+\frac{n}{2}-2}, \ldots, v_{\frac{n}{2}+\lceil\frac{n+2}{4}\rceil}$. So that $v_j \in I[v_i, v_{\frac{n}{2}+\lceil\frac{n+2}{4}\rceil}]$. If $i + \frac{n}{2} \leq j \leq n$, then v_i and v_j will be strongly resolved by v_1 with the shortest path between v_i and v_1 is $v_i, v_{i+\frac{n}{2}}, v_{i+\frac{n}{2}+1}, \ldots, v_n, v_1$. So that $v_j \in I[v_i, v_1]$.

From every possible pairs of vertices, there exists a vertex $s \in S$ which strongly resolves two distinct vertices of $V(M_n) \setminus S$. Thus S is a strong resolving set of Möbius ladder graph M_n.

We obtain a set $S = \{v_1, v_2, \ldots, v_{\lceil\frac{n+2}{4}\rceil}, v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, \ldots, v_{\frac{n}{2}+\lceil\frac{n+2}{4}\rceil}\}$ is a strong resolving set of Möbius ladder graph M_n and we have $|S| \geq 2\left\lceil\frac{n+2}{4}\right\rceil$, so that S is a strong metric basis of M_n. Hence $sdim(M_n) = 2\left\lceil\frac{n+2}{4}\right\rceil$ for n even. □

3. Conclusion

According to the discussion above, we have the strong metric dimension of a sun graph S_n, a windmill graph K_n^m, and a Möbius ladder graph M_n for n even. Furthermore, we conclude this paper with the following open problem for the direction of further research which is still in progress.

Open Problem: Determine the strong metric dimension of a Möbius ladder graph M_n for n odd.

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